Su-Schrieffer-Heeger Model

Lecture Note

September 28, 2026

1. Su-Schrieffer-Heeger (SSH) Model

1.1 Introduction to SSH Model

Consider a 1D polyacetylene. After dimerization, each unit-cell will contain two carbon "atoms" and the bonding between adject carbons "atom" will alternate between single-bonding and double-bonding. If we use $\left(\boldsymbol{a}_j^\dagger, \boldsymbol{a}_j\right)$ and $\left(\boldsymbol{b}_j^\dagger, \boldsymbol{b}_j\right)$ to describe the creation and annihilation operators in site A and B in the j-th unit-cell, then the Hamiltonian of the system is given by

$$\boldsymbol{H} = \sum_{j=-\infty}^{\infty} \left\{ -v\boldsymbol{b}_j^\dagger \boldsymbol{a}_j - w\boldsymbol{a}_{j+1}^\dagger \boldsymbol{b}_j - v\boldsymbol{a}_j^\dagger \boldsymbol{b}_j - w\boldsymbol{b}_j^\dagger \boldsymbol{a}_{j+1} \right\}. \tag{1}$$

Here, v and w denote the intra-cell and inter-cell hopping amplitudes between nearest-neighbor (n.n) sites, respectively. After separation of variables, we have

$$\begin{cases} Ea_j &= -vb_j - wb_{j-1}; \\ Eb_j &= -va_j - wa_{j+1}, \end{cases} \tag{2}$$

with the boundary conditions $b_0 = 0, a_{N+1} = 0$.

Let $a_j = \alpha s^j$ and $b_j = \beta s^j$:

$$\begin{cases} E\alpha = -(v + ws^{-1})\beta; \\ E\beta = -(v + ws)\alpha. \end{cases} \tag{3}$$

To have on-trivial solutions, we have:

$$\begin{vmatrix} E & (v + ws^{-1}) \\ (v + ws) & E \end{vmatrix} = 0 \implies E^2 - (v + ws^{-1})(v + ws) = 0. \tag{4}$$

$$(E^2 - v^2 - w^2) + vw(s + s^{-1}) = 0. \tag{5}$$

The most general solution to the coupled eqation are

$$\begin{cases} a_j = \alpha_+ s^j + \alpha_- s^{-j}; \\ b_j = \beta_+ s^j + \beta_- s^{-j}. \end{cases} \tag{6}$$

(α_+, α_-) and (β_+, β_-) are related through

$$\begin{cases} E\alpha_+ = -(v + ws^{-1})\beta_+; \\ E\beta_+ = -(v + ws)\alpha_+. \end{cases} \quad \text{and} \quad \begin{cases} E\alpha_- = -(v + ws)\beta_-; \\ E\beta_- = -(v + ws^{-1})\alpha_-. \end{cases} \tag{7}$$

Applying the BC $b_0 = 0$ and $a_{N+1} = 0$ and expressing (α_+, α_-) in terms of (β_+, β_-), we have

$$\begin{cases} \beta_+ + \beta_- = 0; \\ (v + ws^{-1})s^{N+1}\beta_+ + (v + ws)s^{-(N+1)}\beta_-. \end{cases} \tag{8}$$

This leads to

$$v\left\{s^{N+1}-s^{-(N+1)}\right\}+w\left\{s^{N}-s^{-N}\right\}=0. \tag{9}$$

After simplification, this is a polynomial of degree $2(N+1)$ in s. Therefore, there should be $2(N+1)$ complex roots. Since there is symmetry in exchanging s and s^{-1}, if s is a root then s^{-1} will also be a root. Although 1 and -1 are roots of the equation, it is obvious that they are not physical. Hence, there are N pairs of roots, which give rise to N-pairs of energy eigenstates.

Let $u=\frac{s+s^{-1}}{2}$ and we have:

$$v\frac{s^{N+1}-s^{-(N+1)}}{s-s^{-1}}+w\frac{s^{N}-s^{-N}}{s-s^{-1}}=0\Rightarrow vU_N(u)+wU_{N-1}(u)=0, \tag{10}$$

where $U_N(u)$ is the Chebyshev polynomial of the second kind:

$$U_N(\cos\theta)=\frac{\sin[(N+1)\theta]}{\sin\theta}, \tag{11}$$

which a polynomiual of degree N. As a result, there are N roots of u. Examples of $U_N(u)$:

$$\begin{aligned}
&U_0(u)=1,\\
&U_1(u)=s+s^{-1}=2u,\\
&U_2(u)=s^2+1+s^{-2}=4u^2-1.
\end{aligned}$$

Consider the case that the N-th unit-cell only contain site A. As a result, the BC become $b_0=0, b_N=0$. Hence, we have $s^{2N}=1 \implies s=e^{i\frac{\pi k}{N}}$, with $k=1,2,\ldots,N-1$. This provides only $2(N-1)$ eigenstates. These states give rise to oscillating wave functions and are the so-called "bulk states."

Zero-Energy Edge States ($E=0$): By counting degree of freedom, we know there is one more eigenstate to be found. Since there is only one state, there should be only one corresponding s so that $a_j=\alpha s^j$, $b_j=\beta s^j$. α,β satisfy

$$\begin{cases}
E\alpha=-(v+ws^{-1})\beta;\\
E\beta=-(v+ws)\alpha.
\end{cases} \tag{12}$$

The BC force β to be zero. To have non-vanishing α, we must have

$$\begin{aligned}
&E=0;\\
&ws+v=0\Rightarrow s=-\frac{v}{w}.
\end{aligned}$$

Therefore,

$$a_j=\alpha\left(-\frac{v}{w}\right)^j,\quad b_j=0 \tag{13}$$

- $v>w\Rightarrow|s|>1$: edge state localized on the right boundary.
- $v<w\Rightarrow|s|<1$: edge state localized on the left boundary.

In the limit that $N\to\infty$, the edge state localized on the right boundary is unphysical, since it wave function blows up at spatial infinity.

- **Trivial Phase ($v>w$):** All solutions satisfy $|s|=1$. There is no topological edge state.
- **Topological Phase ($v<w$):** Solutions with $|s|<1$. There is a topological edge states with $E=0$.

It is obvious that there are two boundaries for a finite chain. We can use the above result of a right semi-infinite SSH chain to have a better understanding of the result of a finite chain. When $v < w$ the chain is in the topological phase. If there are N unit-cells and N is very very large, then we would have an edge state on each boundary. As N decreases, the two edge states would state to mix due to quantum tunneling. It is similar to the bonding and anti-bonding formed between two atoms getting close to each other. The wave functions of them are symmetric and anti-symmetric linear combination of the atomic orbit of the two atoms, respectively. As a result, there is an exponentially small split in energy.

On the other hand, If there are $2N-1$ sites, so that the unit-cell by the right boundary is always different from that by the left boundary. In this case, when the unit-cell by the left boundary is in the topological phase, then the unit-cell by the right boundary would be in the trivial phase, and vice versa. Therefore, there is always only edge state. It would be on the left boundary if $v < w$, and on the right boundary if $v > w$.

Homework: Consider the case that there are N unit-cells and $v < w$ so that the system would have edge states. It is known that the edge state is associated to the root that $s \approx -(v/w)$. Assume $s = -(v/w)+\delta$, and find δ from the equation $v\left\{s^{N+1} - s^{-(N+1)}\right\} + w\left\{s^{N} - s^{-N}\right\} = 0$. Then, use the result to find the approximate form of energy E of the edge state.

1.2 Bloch Hamiltonian and Bulk Edge Correspondence

It is known that we may expand $\boldsymbol{a}_j, \boldsymbol{a}_j^\dagger$ and $\boldsymbol{b}_j, \boldsymbol{b}_j^\dagger$ in the momentum space

$$\begin{cases} \boldsymbol{a}_j = \int_0^{2\pi} \frac{dp}{2\pi} \boldsymbol{a}(p) e^{ipj}, \\ \boldsymbol{a}_j^\dagger = \int_0^{2\pi} \frac{dp}{2\pi} \boldsymbol{a}^\dagger(p) e^{-ipj}. \end{cases} \quad \text{and} \quad \begin{cases} \boldsymbol{b}_j = \int_0^{2\pi} \frac{dp}{2\pi} \boldsymbol{b}(p) e^{ipj}, \\ \boldsymbol{b}_j^\dagger = \int_0^{2\pi} \frac{dp}{2\pi} \boldsymbol{b}^\dagger(p) e^{-ipj}. \end{cases} \tag{14}$$

In terms of $\boldsymbol{a}(p), \boldsymbol{a}^\dagger(p)$ and $\boldsymbol{b}(p), \boldsymbol{b}^\dagger(p)$, the Hamiltonian in eq. (1) may be re-written as

$$\boldsymbol{H} = \int_0^{2\pi} \frac{dp}{2\pi} \left(\boldsymbol{a}^\dagger(p), \boldsymbol{b}^\dagger(p)\right) \mathcal{H}(p) \begin{pmatrix} \boldsymbol{a}(p) \\ \boldsymbol{b}(p) \end{pmatrix}. \tag{15}$$

Here, $\mathcal{H}$ is the Bloch Hamiltonian

$$\mathcal{H}(p) = \begin{pmatrix} 0 & \bar{h}(p) \\ h(p) & 0 \end{pmatrix}, \tag{16}$$

with $h(p) = -(v + we^{ip})$. Solving $\det\{\mathcal{H}(p) - E(p)\mathbf{I}\} = 0 \Rightarrow E^2(p) - |h(p)|^2 = 0$:

$$E(p) = \pm\omega, \tag{17}$$

where $\omega \equiv |h(p)| = \sqrt{v^2 + w^2 + 2vw\cos(p)}$. In the momentum space, the energy eigenstates satisfy the eigen equation

$$\mathcal{H}(p) \mid u_\pm(p) \rangle = \pm\omega \mid u_\pm(p) \rangle. \tag{18}$$

It can be seen that

$$\mid u_\pm(p) \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \pm e^{-i\phi_p} \\ 1 \end{pmatrix}, \tag{19}$$

where $e^{i\phi_p} = h/|h|$.

At the Brillouin zone boundary $p = \pi$:

$$E(\pi) = \pm|v - w| \tag{20}$$

Hence, there is a direct energy band gap between upper and lower bands:

$$\Delta E = 2|v - w|. \tag{21}$$

Therefore, it is an insulator independent of whether v is larger or smaler than w.

When p ranges over the BZ, $h(p)$ trace out a circle centered at $(-v, 0)$ with radius w.

- $v > w$: the winding number ν of $h(p)$ around the origin is zero.
- $v < w$: the winding number ν of $h(p)$ around the origin is one.

Using ν, we can distinguish the two phases and we have the so-called "bulk edge correspondence."

- $v > w, \nu = 0$**, trivial phase**: There is no edge state.
- $v < w, \nu = 1$**, topological phase**: There is a topological edge states with $E = 0$ on the boundary.

We may use the energy eigenstate of the Bloch Hamiltonian to calculate a physical quantity related to the winding number:

$$\langle \boldsymbol{x} \rangle \equiv \langle \Psi \,|\, \boldsymbol{x} \,|\, \Psi \rangle \,. \tag{22}$$

where $\boldsymbol{x}$ is the position operator. From quantum mechanics, we may rewrite the above expression in the momentum space:

$$\langle \boldsymbol{x} \rangle = \int_0^{2\pi} \frac{dp}{2\pi} \left(\tilde{\psi}^\dagger(p) \left\{ i\partial_p \tilde{\psi}(p) \right\} \right). \tag{23}$$

If we use the energy eigenstate in the lower band then we have

$$\langle \boldsymbol{x} \rangle = \frac{1}{2} \int_0^{2\pi} \frac{dp}{2\pi} \left(\partial_p \phi_p \right). \tag{24}$$

It is obvious that $\langle \boldsymbol{x} \rangle$ is proportional to ν. As a result,

- $v > w, \nu = 0, \langle \boldsymbol{x} \rangle = 0$**, trivial phase**: There is no edge state.
- $v < w, \nu = 1, \langle \boldsymbol{x} \rangle = \frac{1}{2}$**, topological phase**: There is a topological edge states with $E = 0$ on the boundary.

The SSH model is the simplest model that exhibits this kind of property. From here, we will make further generalization to higher winding numbers and introduce the extended SSH model. Another generalization is to go to higher dimensions.

Homework: Consider a one-dimensional chain composed of $2N$ particles with alternating spring constants k_a and k_b, and mass m. Let call this the SSH-like coupled spring system. Show that The EOM of such a system is given by

$$\begin{cases} -m\omega^2 a_j + k_a(a_j - b_j) + k_b(a_j - b_{j-1}) = 0, \\ -m\omega^2 b_j + k_a(b_j - a_j) + k_b(b_j - a_{j+1}) = 0. \end{cases} \tag{25}$$

and the BC is given by $b_0 = 0$, $a_{N+1} = 0$.

Homework: Assume that $a_j = \alpha s^j$ and $b_j = \beta s^j$ and show that the resulting secular eqution is given by

$$(-\omega^2 + \omega_a^2 + \omega_b^2)^2 - \left[\omega_a^4 + \omega_b^4 + \omega_a^2\omega_b^2 \left(s + s^{-1}\right)\right] = 0, \tag{26}$$

with $\omega_a^2 = k_a/m$ and $\omega_b^2 = k_b/m$. Show that, by imposing the BC, $b_0 = 0$, $a_{N+1} = 0$, one obtain

$$k_a U_N(u) + k_b U_{N-1}(u) = 0, \tag{27}$$

with $u = \left(s + s^{-1}\right)/2$ and $U_N(u)$ the Chebyshev polynomial of the second kind. Therefore, when the SSH-like coupled spring system satisfies the fixed-end boundary conditions, it also has the "topological phase," when $k_A < k_B$. In this phase, there will be midgap edge states on the boundaries.

The Effect of the Boundary Conditions on the Existence of the Edge States
Let's consider the case that the SSH-like coupled spring system satisfies the free-end boundary conditions: $a_1 - b_0 = 0$ and $a_{N+1} - b_N = 0$. From these boundary conditions, we have

$$\begin{cases} \alpha_+ s + \alpha_- s^{-1} - (\beta_+ + \beta_-) = 0, \\ \alpha_+ s^{N+1} + \alpha_- s^{-N-1} - (\beta_+ s^N + \beta_- s^{-N}) = 0. \end{cases} \tag{28}$$

Introduce $r_\pm = \alpha_\pm / \beta_\pm$, and it may be checked that $r_+ r_- = 1$. Then, we can transform the above equation into

$$\begin{cases} (r_+ s - 1)\beta_+ + (r_- s^{-1} - 1)\beta_- = 0, \\ (r_+ s - 1)\beta_+ s^N + (r_- s^{-1} - 1) s^{-N} \beta_- = 0. \end{cases} \Rightarrow \begin{vmatrix} 1 & 1 \\ s^N & s^{-N} \end{vmatrix} = 0. \tag{29}$$

This leads to $s^{2N} = 1$ and hence $s = e^{i\pi l/N}$.

(i) "Trivial phase": $k_A > k_B$.

$$\begin{cases} \Omega^2 = \sqrt{\omega_A^4 + \omega_B^4 + 2\omega_A^2 \omega_B^2 \cos[\pi l/(N+1)]}, \ l = 1, \ldots, N; \\ \Omega^2 = -\sqrt{\omega_A^4 + \omega_B^4 + 2\omega_A^2 \omega_B^2 \cos[\pi l/(N+1)]}, \ l = 0, \ldots, N-1. \end{cases} \tag{30}$$

Therefore, $\boldsymbol{c}_{2j-1}$ and $\boldsymbol{c}_{2j}$ from the same site j pair up to form the ground state with the occupation number 0.

(b) "Topological phase": $k_A < k_B$.

$$\begin{cases} \Omega^2 = \sqrt{\omega_A^4 + \omega_B^4 + 2\omega_A^2 \omega_B^2 \cos[\pi l/(N+1)]}, \ l = 1, \ldots, N-1; \\ \Omega^2 = -\sqrt{\omega_A^4 + \omega_B^4 + 2\omega_A^2 \omega_B^2 \cos[\pi l/(N+1)]}, \ l = 0, \ldots, N. \end{cases} \tag{31}$$

Note that there is no midgap edge state in this system. This is an explicit example how the boundary conditions of the system affect whether there are edge states on the boundary.

1.3 Hatano-Nelson Model and Non-Hermitian SSH Model

Consider the Hatano-Nelson Model. It is a Non-Hermitian genaralization of the simple tight-binding model: v^R and v^L denote the right and left hopping amplitudes between nearest-neighbor (n.n) sites:

$$\boldsymbol{H} = \sum_{j=-\infty}^{\infty} \left\{ -v^R \boldsymbol{a}_{j+1}^\dagger \boldsymbol{a}_j - v^L \boldsymbol{a}_j^\dagger \boldsymbol{a}_{j+1} \right\}. \tag{32}$$

Non-Hermiticity usually arises when one consider systems with dissipation or open systems. It has become a "hot spot" of research in recent years.

After seperation of variables, the model gives rise to the follwing second order recursive relation:

$$E a_j = -v^R a_{j-1} - v^L a_{j+1}. \tag{33}$$

If we impose the periodic BC, then we set $a_j = \alpha e^{ipj}$ and would obtain

$$E = -\left(v^R e^{-ip} + v^L e^{ip}\right) = -\left(v^R + v^L\right)\cos p + i\left(v^R - v^L\right)\sin p. \tag{34}$$

The energy spectrum is generally complex, which is expected since the system is non-Hermitian.

Consider the case that the system is a finite chain satisfying the open BC. To solve the recursive equation, we let $a_j = \alpha s^j$:

$$E = -\left(v^R s^{-1} + v^L s\right) \Rightarrow s^2 + \frac{E}{v^L} s + \frac{v^R}{v^L} = 0. \tag{35}$$

If we factor the equation into $(s - s_1)(s - s_2)$, we see

$$\begin{cases} s_1 + s_2 = -\frac{E}{v^L}, \\ s_1 \cdot s_2 = \frac{v^R}{v^L}. \end{cases}$$

The most general solution for a_j is:

$$a_j = \alpha_1 s_1^j + \alpha_2 s_2^j \tag{36}$$

If the system satisfies the periodic BC: $a_0 = 0$, $a_{N+1} = 0$, then, we have

$$\begin{cases} \alpha_1 + \alpha_2 = 0, \\ \alpha_1 s_1^{N+1} + \alpha_2 s_2^{N+1} = 0. \end{cases} \Rightarrow \begin{vmatrix} 1 & 1 \\ s_1^{N+1} & s_2^{N+1} \end{vmatrix} = 0. \tag{37}$$

After simplification, we have

$$s_1^{2N} = (s_1 s_2)^N \Rightarrow s_1 = \sqrt{\frac{v^R}{v^L}} e^{ip}, s_2 = \sqrt{\frac{v^R}{v^L}} e^{-ip} \tag{38}$$

with $p = \pi k/(N+1)$ and $k \in \{1, 2, \ldots, N\}$. As a result, we have

$$E = -2v^L \left(\sqrt{\frac{v^R}{v^L}}\right) \cos p. \tag{39}$$

If v^L and v^R are both positive, then the system has a real spectrum. However if v^R is positive, while v^L is negative, then the system has a purely imaginary spectrum. Therefore, when we keep v^R fixed and vary v^L from positive to negative value, we will see a "phase transition" at the value that $v^L = 0$.

In the limit $N \to \infty$, the energy spectrum of the system satisfying the open BC does not apporach that of a system satisfying the periodic BC. Moreover, when $|v^R| < |v^L|$, all the energy eigenstates will be localized on the left boundary and when $|v^R| > |v^L|$, all the energy eigenstates will be localized on the right boundary. This is called the "skin effect". As we will see later, this effect will have a strong effect on the bulk edge correspondence.

Now, let's turn to the non-Hermitian SSH model

$$\boldsymbol{H} = \sum_{j=-\infty}^{\infty} \left\{ -v^R \boldsymbol{b}_j^\dagger \boldsymbol{a}_j - w^R \boldsymbol{a}_{j+1}^\dagger \boldsymbol{b}_j - v^L \boldsymbol{a}_j^\dagger \boldsymbol{b}_j - w^L \boldsymbol{b}_j^\dagger \boldsymbol{a}_{j+1} \right\}. \tag{40}$$

Here, v^R and v^L are the right and left intra-cell hopping amplitudes, while w^R and w^L are the right and left inter-cell hopping amplitudes.

After separation of variables, we have

$$\begin{cases} Ea_j &= -v^L b_j - w^R b_{j-1}; \\ Eb_j &= -v^R a_j - w^L a_{j+1}. \end{cases} \tag{41}$$

If the system satisfie the periodic BC, we let $a_j = \alpha e^{ipj}$ and $b_j = \beta e^{ipj}$. As a result, we have

$$\begin{cases} E\alpha = -(v^L + w^R e^{-ip})\beta; \\ E\beta = -(v^R + w^L e^{ip})\alpha. \end{cases} \tag{42}$$

Therefore, the energy spectrum is given by

$$E = \pm\sqrt{(v^L + w^R e^{-ip})(v^R + w^L e^{ip})}, \tag{43}$$

which is again generally complex.

Next, let consider the case that the system satisfies the open BC $b_0 = 0, a_{N+1} = 0$. Let $a_j = \alpha s^j$ and $b_j = \beta s^j$:

$$\begin{cases} E\alpha = -(v^L + w^R s^{-1})\beta; \\ E\beta = -(v^R + w^L s)\alpha. \end{cases} \tag{44}$$

To have on-trivial solutions, we have:

$$\begin{vmatrix} E & (v^L + w^R s^{-1}) \\ (v^R + w^L s) & E \end{vmatrix} = 0 \Rightarrow E^2 - (v^L + w^R s^{-1})(v^R + w^L s) = 0. \tag{45}$$

After simplification, it takes the following form:

$$s^2 - \frac{(E^2 - w^L w^R - v^L v^R)}{v^L w^L} + \frac{w^R v^R}{v^L w^L} = 0. \tag{46}$$

If s_1 and s_2 are the two roots to the above quadratic equation, we have

$$\begin{cases} s_1 + s_2 = \frac{(E^2 - w^L w^R - v^L v^R)}{v^L w^L}; \\ s_1 s_2 = \frac{w^R v^R}{v^L w^L}. \end{cases} \tag{47}$$

The most general solution to the coupled eqation are

$$\begin{cases} a_j = \alpha_1 s_1^j + \alpha_2 s_2^j; \\ b_j = \beta_1 s_1^j + \beta_2 s_2^j. \end{cases} \tag{48}$$

α_I and β_I is related through

$$\begin{cases} E\alpha_I = -(v + w s_I^{-1})\beta_I; \\ E\beta_I = -(v + w s_I)\alpha_I, \end{cases} \tag{49}$$

for $I = 1, 2$.

Applying the BC $b_0 = 0$ and $a_{N+1} = 0$ and expressing α_I in terms of β_I, we have

$$\begin{cases} \beta_1 + \beta_2 = 0; \\ (v^L + w^R s_1^{-1}) s_1^{N+1} \beta_1 + (v^L + w^R s_2^{-1}) s_2^{(N+1)} \beta_2. \end{cases} \tag{50}$$

This leads to

$$v^L \left\{ s_1^{N+1} - s_2^{(N+1)} \right\} + w^R \left\{ s_1^N - s_2^N \right\} = 0. \tag{51}$$

Making use of the fact that $s_1 s_2 = \frac{w^R v^R}{w^L v^L}$, let's introduce $\tilde{s}$:

$$\begin{cases} s_1 = \sqrt{\frac{w^R v^R}{w^L v^L}}\tilde{s}; \\ s_2 = \sqrt{\frac{w^R v^R}{w^L v^L}}\tilde{s}^{-1}. \end{cases} \tag{52}$$

Define $\bar{v} = \sqrt{v^L v^R}$ and $\bar{w} = \sqrt{w^L w^R}$, we now have

$$E^2 = \bar{v}^2 + \bar{w}^2 + 2\bar{v}\bar{w}\tilde{u}, \tag{53}$$

$$\bar{v} U_N(\tilde{u}) + \bar{w} U_{N-1}(\tilde{u}). \tag{54}$$

Here $\tilde{u} = \left(\tilde{s} + \tilde{s}^{-1}\right)/2$. If all the hopping amplitudes are positive, the results are exactly the same as the case of the Hermitian SSH model with $(\bar{v}, \bar{w})$ playing the role of (v, w). From

what we learned in the Hermitian SSH model, the system would be in the topological phase when $\bar{w} > \bar{v}$. When this is the case, the $\tilde{s}$ corresponding to the edge states are given by $\tilde{s} \approx -(\bar{v}/\bar{w}), -(\bar{w}/\bar{v})$. This translates to $(s_1, s_2) \approx (-v^R/w^L, -w^R/v^L)$. Therefore, both edge states will be localized on the left boundary if $v^R < w^L$ and $w^R < v^L$. In contrast, both edge states will be localized on the right boundary if $v^R > w^L$ and $w^R > v^L$. This is clearly in violation of the bulk edge correspondence we see in the Hermitian SSH model.

We can intruduce the so-called spectrum winding number ν_E to classify the non-Hermitian SSH model. Here, ν_E is the winding number of E^2 (satisfying the periodic PC) with respect to the origin on the complex plane. From eq. (43), we see that when $v^R < w^L$ and $w^R < v^L$ $\nu_E = 1$. On the other hand when $v^R > w^L$ and $w^R > v^L$ $\nu_E = -1$. Let's summarize the results here:

- $\bar{v} > \bar{w}, \nu = 0$**, trivial phase**: There is no edge state.
- $\bar{v} < \bar{w}, \nu = 1$**, topological phase**: Two edge states.
  - (i) $v^R < w^L$**,** $w^R > v^L$**,** $\nu_E = 0$: There are one left edge state and one right edge state.
  - (ii) $v^R < w^L$**,** $w^R < v^L$**,** $\nu_E = 1$: There are two left edge states.
  - (iii) $v^R > w^L$**,** $w^R > v^L$**,** $\nu_E = -1$: There are two right edge states.

1.4 Extended SSH Models

Let's now consider hoppings beyond the n.n. We will call this kind of models the extended SSH models. In particular, we will first consider the following type A extended SSH model:

$$\boldsymbol{H} = \sum_{j=-\infty}^{\infty} \left\{ -t_0 \boldsymbol{b}_j^\dagger \boldsymbol{a}_j - t_1 \boldsymbol{a}_{j+1}^\dagger \boldsymbol{b}_j - t_2 \boldsymbol{a}_{j+2}^\dagger \boldsymbol{b}_j - t_0 \boldsymbol{a}_j^\dagger \boldsymbol{b}_j - t_1 \boldsymbol{b}_j^\dagger \boldsymbol{a}_{j+1} - t_2 \boldsymbol{b}_j^\dagger \boldsymbol{a}_{j+2} \right\}. \tag{55}$$

Here, we replace v and w by t_0 and t_1, repectively. The newly introduced hopping amplitude t_2 describe a hopping amplitude beyond n.n hopping. After separation of variables, we have

$$\begin{cases} Ea_j &= -t_0 b_j - t_1 b_{j-1} - t_2 b_{j-2}; \\ Eb_j &= -t_0 a_j - t_1 a_{j+1} - t_2 a_{j+2}. \end{cases} \tag{56}$$

In the momentum space, the Bloch Hamiltonian $\mathcal{H}$ is again given by

$$\mathcal{H}(p) = \begin{pmatrix} 0 & \bar{h}(p) \\ h(p) & 0 \end{pmatrix}, \tag{57}$$

with $h(p) = -\left(t_0 + t_1 e^{ip} + t_2 e^{i2p}\right)$. It is easy to see that

$$h(p) = -e^{ip}\left(t_0 e^{-ip} + t_1 + t_2 e^{ip}\right) = -e^{ip}\left\{t_1 + (t_0 + t_2)\cos p + i(t_2 - t_0)\sin p\right\}. \tag{58}$$

Since $h(p)$ may be factored into two terms, the total winding number of $h(p)$ is just the sum of them. It is obvious that the factor $-e^{ip}$ always contributes 1 to the winding number, since it traces out a unit circle counter-clockwise in the complex plane. Meanwhile, the second factor traces out a ellipse centered at $(t_1, 0)$. The semi-major is $t_2 + t_0$, while the semi-minor is $|t_2 - t_0|$. As a result, if $t_2 + t_0 < t_1$, the ellipse does not wrap around the origin and the second term's contribution to the winding number is zero. If $t_2 + t_0 > t_1$ and $t_2 - t_0 > 0$, its contribution to the winding number is one . In contrast, its contribution to the winding number is minus one if $t_2 + t_0 > t_1$ and $t_2 - t_0 < 0$. In summary,

- $t_2 + t_0 > t_1, t_2 - t_0 < 0, \nu = 0$, **trivial phase**: There is no edge state.

- $t_2 + t_0 < t_1, \nu = 1$, **topological phase**: One edge state on the boundary.
- $t_2 + t_0 > t_1, t_2 - t_0 > 0, \nu = 2$, **topological phase**: Two edge states on the boundary.

Analogously, we may consider the following type B extended SSH model:

$$\boldsymbol{H} = \sum_{j=-\infty}^{\infty} \Big\{ -t_0 \boldsymbol{b}_j^\dagger \boldsymbol{a}_j - t_1 \boldsymbol{a}_{j+1}^\dagger \boldsymbol{b}_j - t_{-1} \boldsymbol{a}_{j-1}^\dagger \boldsymbol{b}_j - t_0 \boldsymbol{a}_j^\dagger \boldsymbol{b}_j - t_1 \boldsymbol{b}_j^\dagger \boldsymbol{a}_{j+1} - t_{-1} \boldsymbol{b}_j^\dagger \boldsymbol{a}_{j-1} \Big\}. \tag{59}$$

Here, the hopping amplitude t_{-1} describe another type of hopping amplitude beyond n.n hopping. After separation of variables, we have

$$\begin{cases} Ea_j &= -t_0 b_j - t_1 b_{j-1} - t_{-1} b_{j+1}; \\ Eb_j &= -t_0 a_j - t_1 a_{j+1} - t_{-1} a_{j-1}. \end{cases} \tag{60}$$

In the momentum space, the Bloch Hamiltonian $\mathcal{H}$ is again given by

$$\mathcal{H}(p) = \begin{pmatrix} 0 & \bar{h}(p) \\ h(p) & 0 \end{pmatrix}, \tag{61}$$

with $h(p) = -\left(t_0 + t_1 e^{ip} + t_{-1} e^{-ip}\right)$. In summary,

- $t_1 + t_{-1} > t_0, t_1 - t_{-1} < 0, \nu = -1$, **topological phase**: One edge state on the boundary.
- $t_1 + t_{-1} < t_0, \nu = 0$, **trivial phase**: There is no edge state.
- $t_1 + t_{-1} > t_0, t_1 - t_{-1} > 0, \nu = 2$, **topological phase**: One edge state on the boundary.

Note that If we define $(\tilde{\boldsymbol{a}}_j, \tilde{\boldsymbol{b}}_j) = (\boldsymbol{b}_j, \boldsymbol{a}_{j+1})$ and $(\tilde{\boldsymbol{a}}_j^\dagger, \tilde{\boldsymbol{b}}_j^\dagger) = (\boldsymbol{b}_j^\dagger, \boldsymbol{a}_{j+1}^\dagger)$, we may transform a type B extended model into a type A extended model and vice versa.

Homework: Show that for a right semi-infinite type A extended SSH model, the open BC is given by $b_0 = 0, b_{-1} = 0$. From this BC find out the conditions that (i) there are two edge states, (ii) there is a edge state. (iii) there is no edge state. Show that these conditions is equivalent to those obtained using the winding number of the system.

1.5 2D Tight-Binding Model

For a 2D square lattice with hopping parameter v:

$$i\frac{dA_{j_1,j_2}}{dt} = -v(A_{j_1+1,j_2} + A_{j_1-1,j_2} + A_{j_1,j_2+1} + A_{j_1,j_2-1}). \tag{62}$$

After separation of variables, we have

$$Ea_{j_1,j_2} = -v(a_{j_1+1,j_2} + a_{j_1-1,j_2} + a_{j_1,j_2+1} + a_{j_1,j_2-1}). \tag{63}$$

If the periodic BC is satisfied, we would have

$$E = -2v\{\cos p_1 + \cos p_2\}. \tag{64}$$

Here, $p_1 = 2\pi k_1/N_1$, $p_1 = 2\pi k_2/N_2$, with k_1 N_1 consecutive integers and k_2 N_2 consecutive integers.

Homework: Find the energy spectrum for a 2D tight-binding model which satisfies the open BC. Check the result with numerical calculation.