Other Topological Materials

Lecture Note

November 22, 2026

1. Topological Superconductor

1.1 Kitaev Chain

Let's consider the Kitaev chain, which describes a 1D topological superconductor:

$$\boldsymbol{H}_{\mathrm{K}} = \sum_{j=-\infty}^{\infty} \left\{ -\mu \left(\boldsymbol{a}_j^\dagger \boldsymbol{a}_j - \frac{1}{2} \right) - w\, \boldsymbol{a}_{j+1}^\dagger \boldsymbol{a}_j - w \boldsymbol{a}_j^\dagger \boldsymbol{a}_{j+1} + \Delta\, \boldsymbol{a}_j \boldsymbol{a}_{j+1} + \Delta\, \boldsymbol{a}_j^\dagger \boldsymbol{a}_{j+1}^\dagger \right\}, \tag{1}$$

with Δ being the p-wave superconductivity pairing. Since $\boldsymbol{a}_j^\dagger$ and $\boldsymbol{a}_j$ are fermionic creation and annihilation operators, they satisfy the cannonical anti-commutation relation:

$$\boldsymbol{a}_i^\dagger \boldsymbol{a}_j + \boldsymbol{a}_j \boldsymbol{a}_j^\dagger = \delta_{ij}. \tag{2}$$

In the momentum space, the Hamiltonian is given by

$$\boldsymbol{H}_{\mathrm{K}} = \int_0^{2\pi} \frac{dp}{2\pi} \left(\boldsymbol{a}^\dagger(p), \boldsymbol{a}(-p) \right) \mathcal{H}(p) \begin{pmatrix} \boldsymbol{a}(p) \\ \boldsymbol{a}^\dagger(-p) \end{pmatrix}. \tag{3}$$

Here, $\mathcal{H}$ is the Bloch Hamiltonian

$$\mathcal{H}(p) = h_3(p)\sigma_3 + h_2(p)\sigma_2 = \begin{pmatrix} -\mu - 2w\cos p & 2i\Delta \sin p \\ -2i\Delta \sin p & \mu + 2w\cos p \end{pmatrix}. \tag{4}$$

Here, $h_3(p) = -\mu - 2w\cos p$ and $h_2(p) = -2\Delta \sin p$. The energy spectrum of the system is given by

$$E(p) = \pm\sqrt{(\mu + 2w\cos p)^2 + 4\Delta^2 \sin^2 p}. \tag{5}$$

By rotation along the y-axis in the internal space, we can rotate the z-axis into the x-axis. As a result, the rotated Bloch Hamiltonian becomes

$$\tilde{\mathcal{H}}(p) = \begin{pmatrix} 0 & \bar{h}(p) \\ h(p) & 0 \end{pmatrix}. \tag{6}$$

Here, $h(p) = -\mu - 2w\cos p - 2i\Delta \sin p$ and $\bar{h}(p)$ the complex conjugate of $h(p)$. Now, it is clear that $h(p)$ trace out a ellipse in the complex plane of $h(p)$. The center of the ellipse is at $(-\mu, 0)$. The semi-major and semi-minor axes are $2|w|$ and $2|\Delta|$, respectively, when $|w| > |\Delta|$. In contrast, when $|w| < |\Delta|$, the semi-major and semi-minor axes become $2|\Delta|$ and $2|w|$, respectively. However, independent of wheter $|w|$ or $|\Delta|$ is lager, the winding number of the system is non-vanishing when $2|w| > |\mu|$. In such a case, the Kitaev chain would be in the topological phase and zero-energy edge states would appear on the boundary, just like in the case of the SSH model. Of course, in this case the chain must statisfy the open boundary conditions.

To be concrete, let's write down the Schrodinger equation in the position space:

$$i\dot{A}(j,t) + \mu A(j,t) + w\Big\{A(j+1,t) + A(j-1,t)\Big\} + \Delta\Big\{\bar{A}(j+1,t) - \bar{A}(j-1,t)\Big\} = 0, \qquad (7)$$

for $2 \leq j \leq N-1$. Define $\bar{b}_j$ to be the complex conjugate of b_j. For separartion of variables, we let $A(j,t) = e^{-iEt}a_j + e^{i\bar{E}t}\bar{b}_j$ and $\bar{A}(j,t) = e^{iEt}\bar{a}_j + e^{-iEt}b_j$. As aresult, we obatin

$$\begin{aligned}
&(E+\mu)a_j + w\{a_{j+1} + a_{j-1}\} + \Delta\{b_{j+1} - b_{j-1}\} = 0, \\
&(-E+\mu)b_j + w\{b_{j+1} + b_{j-1}\} + \Delta\{a_{j+1} - a_{j-1}\} = 0.
\end{aligned} \qquad (8)$$

To solve the above difference equation, we let $a_j = \alpha s^j$ and $b_j = \beta s^j$ and we have

$$\begin{aligned}
&\Big\{(E+\mu)\alpha + w(s+s^{-1})\Big\} + \Delta\Big\{s - s^{-1}\Big\}\beta = 0, \\
&\Delta\Big\{s - s^{-1}\Big\}\alpha + \Big\{(-E+\mu) + w(s+s^{-1})\Big\}\beta = 0.
\end{aligned} \qquad (9)$$

Again, the determinant of the 2×2 matrix formed by the coefficients of α and β must be vanishing:

$$E^2 - \Big\{\mu + w(s+s^{-1})\Big\}^2 - \Delta^2\Big\{s - s^{-1}\Big\}^2 = 0. \qquad (10)$$

This is a quartic equation in s and it is obvious that if s is root so will be s^{-1}. The most general form of the solution is given by

$$\begin{aligned}
a_j &= \sum_{I=1,2}\Big\{\alpha_{I,+}s_I^j + \alpha_{I,-}s_I^{-j}\Big\}, \\
b_j &= \sum_{I=1,2}\Big\{\beta_{I,+}s_I^j + \beta_{I,-}s_I^{-j}\Big\}.
\end{aligned} \qquad (11)$$

It is understood that $\alpha_{I,+}$ and $\beta_{I,+}$ are related through eq. (9). The open boundary conditions are given by $a_0 = 0, b_0 = 0$ and $a_{N+1} = 0, b_{N+1} = 0$ and we have

$$\begin{cases}
\sum_{I=1,2}\Big\{\alpha_{I,+} + \alpha_{I,-}\Big\} = 0, \\
\sum_{I=1,2}\Big\{\beta_{I,+} + \beta_{I,-}\Big\} = 0, \\
\sum_{I=1,2}\Big\{\alpha_{I,+}s_I^{N+1} + \alpha_{I,-}s_I^{-N-1}\Big\} = 0, \\
\sum_{I=1,2}\Big\{\beta_{I,+}s_I^{N+1} + \beta_{I,-}s_I^{-N-1}\Big\} = 0.
\end{cases} \qquad (12)$$

Using eq. (9), we can express and $\beta_{I,\pm}$ in terms of $\alpha_{I,\pm}$. Then, by requiring the determinant of the 4×4 matrix formed by the coefficients of $\alpha_{+,1}, \alpha_{+,2}, \alpha_{-,1}, \alpha_{-,2}$ to be vanishing, we obatin a secular eqution in terms of s_1 and s_2. Since $s_1, s_2, s_1^{-1}, s_2^{-1}$ are the roots of the quartic equation of s given in eq. (10), their relation can be found through the coefficients of the equation. Similarly, we can express E in terms of s_1 and s_2 through the coefficients of the equation in eq. (10).

Kitaev introduces the so-called Majorana operators:

$$\boldsymbol{c}_{2j-1} = \boldsymbol{a}_j + \boldsymbol{a}_j^\dagger, \quad \boldsymbol{c}_{2j} = \frac{\boldsymbol{a}_j + \boldsymbol{a}_j^\dagger}{i}, \; j = 1, 2, \ldots, N. \qquad (13)$$

Note that the Majorana operators are analogous to real fields, since

$$\boldsymbol{c}_m^\dagger = \boldsymbol{c}_m, \; m = 1, 2, \ldots, 2N. \qquad (14)$$

In other words, they are their own anti-particles. They satisfy the following anti-commutation relation:

$$\boldsymbol{c}_l\boldsymbol{c}_m + \boldsymbol{c}_m\boldsymbol{c}_l = 2\delta_{lm},\ l, m = 1, 2, \ldots, 2N. \tag{15}$$

Since $\boldsymbol{c}_{2j-1}$ and $\boldsymbol{c}_{2j}$ belongs to different sites, the "phase error"

$$\boldsymbol{a}_j^\dagger \boldsymbol{a}_j = \frac{1}{2}\Big(1 + 2i\boldsymbol{c}_{2j-1}\boldsymbol{c}_{2j}\Big), \tag{16}$$

would be unlikely to occur.

Indeed, it would require interaction between the two 'Majorana sites' which could be possibly avoided. Note that a single Majorana operator $\boldsymbol{c}_{2j-1}$ or $\boldsymbol{c}_{2j}$ can not appear as a single term in any reasonable Hamiltonian because it does not preserve the fermionic parity. Thus an isolated Majorana site (usually called a Majorana fermion) is immune to any kind of error!

In terms of these operators, the Hamiltonian is given by

$$\boldsymbol{H}_{\rm K} = \frac{i}{2}\sum_j \Big\{-\mu\ (\boldsymbol{c}_{2j-1}\boldsymbol{c}_{2j}) + (w+\Delta)\boldsymbol{c}_{2j+1}\boldsymbol{c}_{2j} + (-w+\Delta)\boldsymbol{c}_{2j-1}\boldsymbol{c}_{2j+2}\Big\}, \tag{17}$$

It is known that the trivial and topological phases are separated by the line $|\mu| = 2|w|$ and this is indpendent of the value of Δ. Let's choose two special cases to simplify our analysis of a finite chain.

(i) Trivial phase: $w = \Delta = 0$, $\mu < 0$.

$$\boldsymbol{H}_{\rm K} = \frac{-i\mu}{2}\sum_{j=1}^{N}\Big\{\boldsymbol{c}_{2j-1}\boldsymbol{c}_{2j}\Big\}. \tag{18}$$

Therefore, $\boldsymbol{c}_{2j-1}$ and $\boldsymbol{c}_{2j}$ from the same site j pair up to form the ground state with the occupation number 0.

(b) Topological phase phase: $\Delta = w > 0$, $\mu = 0$.
In this case,

$$\boldsymbol{H}_{\rm K} = iw\sum_{j=1}^{N-1}\Big\{\boldsymbol{c}_{2j+1}\boldsymbol{c}_{2j}\Big\}. \tag{19}$$

Now, $\boldsymbol{c}_{2j+1}$ and $\boldsymbol{c}_{2j}$ from different sites j and $j+1$ are paired up to form the ground state. We can define new annihilation and creation operators:

$$\tilde{\boldsymbol{a}}_j = \frac{\boldsymbol{c}_{2j} + i\boldsymbol{c}_{2j+1}}{2}, \quad \tilde{\boldsymbol{a}}_j^\dagger = \frac{\boldsymbol{c}_{2j} - i\boldsymbol{c}_{2j+1}}{2},\ j = 1, 2, \ldots, N-1. \tag{20}$$

with $\boldsymbol{c}_1$ and $\boldsymbol{c}_{2N}$ un-paired. In terms of $\tilde{\boldsymbol{a}}_j$ and $\tilde{\boldsymbol{a}}_j^\dagger$, the Hamiltonian becomes

$$\boldsymbol{H}_{\rm K} = 2w\sum_{j=1}^{N-1}\Big\{\tilde{\boldsymbol{a}}_j^\dagger\tilde{\boldsymbol{a}}_j - \frac{1}{2}\Big\}. \tag{21}$$

The ground state of the system must satisfy the condition:

$$\tilde{\boldsymbol{a}}_j \mid \psi\ \rangle = 0, \quad \text{for } j = 1, \ldots, N-1. \tag{22}$$

In fact, there are two orthogonal ground states $\mid \psi_0\ \rangle$ and $\mid \psi_1\ \rangle$ with this property. The two Majorana operators $\boldsymbol{b}' = \boldsymbol{c}_1$ and $\boldsymbol{b}'' = \boldsymbol{c}_{2N}$ do not show up in the Hamiltonian. Therefore,

they commute with the Hamiltonian. Since $(\boldsymbol{b}')^2 = (\boldsymbol{b}'')^2 = 1$ and $\boldsymbol{b}'\boldsymbol{b}'' = -\boldsymbol{b}''\boldsymbol{b}'$, we see that $(i\boldsymbol{b}'\boldsymbol{b}'')^2 = 1$. Hence, we can define two projection operators:

$$\boldsymbol{P}_0 = \frac{1}{2}(1 + i\boldsymbol{b}'\boldsymbol{b}''), \quad \boldsymbol{P}_1 = \frac{1}{2}(1 - i\boldsymbol{b}'\boldsymbol{b}''). \tag{23}$$

It can be shown that they satisfy $(\boldsymbol{P}_0)^2 = (\boldsymbol{P}_1)^2 = 1$ and $\boldsymbol{P}_0 + \boldsymbol{P}_1 = 1$. Moreover,

$$(i\boldsymbol{b}'\boldsymbol{b}'')\boldsymbol{P}_0 = \boldsymbol{P}_0, \quad (i\boldsymbol{b}'\boldsymbol{b}'')\boldsymbol{P}_1 = -\boldsymbol{P}_1. \tag{24}$$

Using these projection operators, we can construct the two ground states previously mentioned and they obey

$$\begin{aligned} &\boldsymbol{P}_0 \,|\, \psi_0 \,\rangle = |\, \psi_0 \,\rangle\,, \quad \boldsymbol{P}_1 \,|\, \psi_1 \,\rangle = |\, \psi_1 \,\rangle\,; \\ &(i\boldsymbol{b}'\boldsymbol{b}'') \,|\, \psi_0 \,\rangle = |\, \psi_0 \,\rangle\,, \quad (i\boldsymbol{b}'\boldsymbol{b}'') \,|\, \psi_1 \,\rangle = -\,|\, \psi_1 \,\rangle\,. \end{aligned} \tag{25}$$

Therefore, $|\, \psi_0 \,\rangle$ and $|\, \psi_1 \,\rangle$ have even and odd number of electrons, respectively.

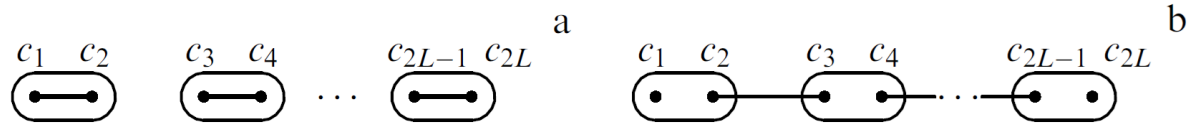


Figure 2. Two types of pairing.

Figure 1: Two type of pairings for the trivial and topological phases.

In principle, one may use the proximity effect of superconductors to relaize p-wave superconductivity.