Qi-Wu-Zhang Model

Lecture Note

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1. Rice-Mele (RM) Model

1.1 Introduction to RM Model

Let's generalize the SSH model to the so-called Rice-Mele (RM) model by including a "chemical potential" term

$$\boldsymbol{H} = \sum_{j=-\infty}^{\infty} \Big\{ u_0 \boldsymbol{a}_j^\dagger \boldsymbol{a}_j - u_0 \boldsymbol{b}_j^\dagger \boldsymbol{b}_j - v \boldsymbol{b}_j^\dagger \boldsymbol{a}_j - w \boldsymbol{a}_{j+1}^\dagger \boldsymbol{b}_j - v \boldsymbol{a}_j^\dagger \boldsymbol{b}_j - w \boldsymbol{b}_j^\dagger \boldsymbol{a}_{j+1} \Big\}. \tag{1}$$

Here, site A has chemical potential u_0 and site B has chemical potential $-u_0$. After separation of variables, we have

$$\begin{cases} Ea_j &= u_0 a_j - v b_j - w b_{j-1}; \\ Eb_j &= -u_0 b_j - v a_j - w a_{j+1}, \end{cases} \tag{2}$$

with the boundary conditions $b_0 = 0, a_{N+1} = 0$.

Let $a_j = \alpha s^j$ and $b_j = \beta s^j$:

$$\begin{cases} (E - u_0)\alpha = -(v + ws^{-1})\beta; \\ (E + u_0)\beta = -(v + ws)\alpha. \end{cases} \tag{3}$$

To have on-trivial solutions, we have:

$$\begin{vmatrix} E - u & (v + ws^{-1}) \\ (v + ws) & E + u \end{vmatrix} = 0 \implies E^2 - u_0^2 - (v + ws^{-1})(v + ws) = 0. \tag{4}$$

$$(E^2 - u_0^2 - v^2 - w^2) + vw(s + s^{-1}) = 0. \tag{5}$$

The most general solution to the coupled eqation are

$$\begin{cases} a_j = \alpha_+ s^j + \alpha_- s^{-j}; \\ b_j = \beta_+ s^j + \beta_- s^{-j}. \end{cases} \tag{6}$$

(α_+, α_-) and (β_+, β_-) are related through

$$\begin{cases} (E - u_0)\alpha_+ = -(v + ws^{-1})\beta_+; \\ (E + u_0)\beta_+ = -(v + ws)\alpha_+. \end{cases} \quad \text{and} \quad \begin{cases} (E - u_0)\alpha_- = -(v + ws)\beta_-; \\ (E + u_0)\beta_- = -(v + ws^{-1})\alpha_-. \end{cases} \tag{7}$$

Applying the BC $b_0 = 0$ and $a_{N+1} = 0$ and expressing (α_+, α_-) in terms of (β_+, β_-), we have

$$\begin{cases} \beta_+ + \beta_- = 0; \\ (v + ws^{-1})s^{N+1}\beta_+ + (v + ws)s^{-(N+1)}\beta_-. \end{cases} \tag{8}$$

This again leads to the same equation as that of the SSH model:

$$vU_N(u) + wU_{N-1}(u) = 0. \tag{9}$$

Here, $u = \frac{s+s^{-1}}{2}$, and $U_N(u)$ is the Chebyshev polynomial of the second kind. Similar to the SSH model case

- $v > w$: no edge state.
- $v < w$: one edge state with energy near $-u_0$ localized on the left boundary and one edge state with energy near u_0 localized on the right boundary.

Homework: Consider the case that there are N unit-cells and $v < w$ so that the system would have edge states, similar to the case of the SSH model. It is also known that the edge state is associated to the root that $s \approx -(v/w)$. Assume $s = -(v/w) + \delta$, and find δ from the equation $v\left\{s^{N+1} - s^{-(N+1)}\right\} + w\left\{s^N - s^{-N}\right\} = 0$. Then, use the result to find the approximate form of energy E of the edge state. Verify that the edge state with energy near $-u_0$ is localized on the left boundary and the edge state with energy near u_0 is localized on the right boundary.

1.2 Bloch Hamiltonian and Chiral Symmetry

Similar to the SSH model the Hamiltonian in eq. (1) may be re-written in terms of $\boldsymbol{a}(p), \boldsymbol{a}^\dagger(p)$ and $\boldsymbol{b}(p), \boldsymbol{b}^\dagger(p)$,

$$\boldsymbol{H} = \int_0^{2\pi} \frac{dp}{2\pi} \left(\boldsymbol{a}^\dagger(p), \boldsymbol{b}^\dagger(p)\right) \mathcal{H}(p) \begin{pmatrix} \boldsymbol{a}(p) \\ \boldsymbol{b}(p) \end{pmatrix}. \tag{10}$$

Here, the Bloch Hamiltonian $\mathcal{H}$ is given by

$$\mathcal{H}(p) = \begin{pmatrix} u_0 & \bar{h}(p) \\ h(p) & -u_0 \end{pmatrix}, \tag{11}$$

with $h(p) = -(v + we^{ip})$. Solving $\det\{\mathcal{H}(p) - E(p)\mathbf{I}\} = 0 \Rightarrow E^2(p) - |h(p)|^2 = 0$:

$$E(p) = \pm\omega, \tag{12}$$

where $\omega = \sqrt{u_0^2 + |h|^2} = \sqrt{u_0^2 + v^2 + w^2 + 2vw\cos(p)}$. In the momentum space, the energy eigenstates satisfy the eigen equation

$$\mathcal{H}(p) \mid \psi_\pm(p) \rangle = \pm\omega \mid \psi_\pm(p) \rangle. \tag{13}$$

It can be seen that

$$\mid \psi_\pm(p) \rangle = \frac{1}{\sqrt{2\omega(\omega \mp u_0)}} \begin{pmatrix} \pm\bar{h} \\ \omega \mp u_0 \end{pmatrix}. \tag{14}$$

Remember that in the SSH model $\langle \boldsymbol{x} \rangle$ is proportional to ν and in the momentum space,

$$\langle \boldsymbol{x} \rangle = \int_0^{2\pi} \frac{dp}{2\pi} \left(\tilde{\psi}^\dagger(p) \left\{ i\partial_p \tilde{\psi}(p) \right\} \right). \tag{15}$$

Moreover, $\langle \boldsymbol{x} \rangle$ is quantized, either 0 or 1/2. In the RM model, however, this not the case, and the system no longer has a topological phase. The main reason is that the chiral symmetry in the SSH model is now broken in the RM model. For the SSH, the the Bloch Hamiltonian may be expressed as

$$\mathcal{H} = h_1(p)\sigma_1 + h_2(p)\sigma_2. \tag{16}$$

Here, $h_1(p)$ and $h_2(p)$ are the real and imaginary part of $h(p)$, while σ_1 and σ_2 are the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Since σ_3 anti-commutes with both σ_1 and σ_2 so that $\{\sigma_3, \mathcal{H}\} = 0$, it can serve as a chiral symmetry of the system. If $| \psi_E(p) \rangle$ is an energy eigenstate with eigenvalue $E(p)$

$$\mathcal{H}(p) | \psi_E(p) \rangle = E(p) | \psi_E(p) \rangle,$$

then

$$\mathcal{H}(p) \{\sigma_3 | \psi_E(p) \rangle\} = -\sigma_3 \{\mathcal{H}(p) | \psi_E(p) \rangle\} = -E(p) \{\sigma_3 | \psi_E(p) \rangle\}. \tag{17}$$

Therefore, we see $\sigma_3 | u_E(p) \rangle$ would be an energy eigenstate with eigenvalue $-E(p)$. In conclusion, when there is a chiral symmetry in the system, non-zero energy eigenstates always appear in pairs.

For the RM model, the the Bloch Hamiltonian is gven by

$$\mathcal{H} = h_1(p)\sigma_1 + h_2(p)\sigma_2 + u_0\sigma_3. \tag{18}$$

Hence, the sytem no longer has the chiral symmetry. Note that the Bloch Hamiltonian

$$\mathcal{H} = h_1(p)\sigma_1 + h_2(p)\sigma_2 + h_3(p)\sigma_3, \tag{19}$$

with $h_1(p), h_2(p), h_3(p)$ all real, is the most general form of Hamiltonians that a two-band model can have.

2. Adiabatic Approximation and the Berry Phase

Consider a system with a time-dependent Hamiltonian, $\boldsymbol{H}(t)$, so that we may define the so-called instantaneous energy eigenstates:

$$\boldsymbol{H}(t) | \psi_n(t) \rangle = E_n(t) | \psi_n(t) \rangle. \tag{20}$$

For simplicity, let's assume that the system is non-degenerate. It is known that the time evolution of a quantum state is governed by the Schrodinger equation:

$$i\partial_t | \Psi(t) \rangle = \boldsymbol{H}(t) | \Psi(t) \rangle. \tag{21}$$

Define ΔE to be the minimal energy difference among the states that we are interested in, one may define a time scale $T_0 = \hbar/\Delta E$ of the system. When the condition $|E_n(t)/\dot{E}_n(t)| \gg T_0$ is satisfied, the adiabatic theorem may be applied to the system: if the initial state is the n-th instantaneous energy eigenstate at time $t = 0$, i.e. $| \Psi(0) \rangle = | \psi_n(0) \rangle$, then the time-evolved state, $| \Psi(t) \rangle$, will remain in the n-th instantaneous energy eigenstate at time t:

$$| \Psi(t) \rangle \propto | \psi_n(t) \rangle = e^{i\gamma_n(t)} e^{-i\theta_n(t)} | \psi_n(t) \rangle. \tag{22}$$

Here, $\theta_n(t) = \int_0^t E_n(t')dt'$ is the dynamical phase and $\gamma_n(t)$ is the geometric phase. Substitute the above relation into eq. (21), and we have

$$e^{i\gamma_n(t)} e^{-i\theta_n(t)} \Big(\big\{ -\dot{\gamma}_n(t) + E_n(t) \big\} | \Psi(t) \rangle + i\partial_t | \Psi(t) \rangle \Big) = e^{i\gamma_n(t)} e^{-i\theta_n(t)} \big\{ E_n(t) \big\} | \psi_n(t) \rangle. \tag{23}$$

Since $\langle \psi_n(t) | \psi_n(t) \rangle = 1$, the above eqaution may be simplified to

$$\dot{\gamma}_n(t) = i \langle \psi_n(t) | \partial_t \psi_n(t) \rangle. \tag{24}$$

If $\boldsymbol{H}(t)$ is periodic in t with period T, then $\mid \Psi(T) \rangle$ would go back to the initial state $\mid \psi_n(0) \rangle$, with an extra phase $\gamma(T)$ in addition to the dynamical phase $\theta_n(T)$:

$$\gamma_n(T) = i \int_0^T dt \langle \psi_n(t) \mid \partial_t \psi_n(t) \rangle , \tag{25}$$

which is called the Berry phase.

To calculate the Berry phase numerically, we can discretize the time by introducing a large natural number N. Let's define $\Delta = T/N$ and $t_j = j * \Delta$, with $j = 0, 1, 2, \ldots, N$. Now, we will consider the following inner product:

$$\langle \psi_n(t_j) \mid \psi_n(t_{j+1}) \rangle . = \langle \psi_n(t_j) \mid \psi_n(t_j + \Delta) \rangle . \tag{26}$$

If Δ is small enough so that we may expand $\mid \psi_n(t_j + \Delta) \rangle \approx \mid \psi_n(t_j) \rangle + \Delta \mid \partial_t \psi_n(t_j) \rangle$, then the previous inner product simplifies to

$$\langle \psi_n(t_j) \mid \psi_n(t_j + \Delta) \rangle \approx 1 + \Delta \langle \psi_n(t_j) \mid \partial_t \psi_n(t_j) \rangle \approx e^{-i\Delta\dot{\gamma}_n(t_j)}. \tag{27}$$

Now, let's consider the product

$$\prod_{j=0}^{N-1} \langle \psi_n(t_j) \mid \psi_n(t_{j+1}) \rangle = \langle \psi_n(t_0) \mid \psi_n(t_1) \rangle \langle \psi_n(t_1) \mid \psi_n(t_2) \rangle \ldots \langle \psi_n(t_{N-1}) \mid \psi_n(t_N) \rangle . \tag{28}$$

It is not difficult to see that

$$\prod_{j=0}^{N-1} \langle \psi_n(t_j) \mid \psi_n(t_{j+1}) \rangle \approx \exp\left\{ -i \sum_{j=0}^{N-1} \Delta\dot{\gamma}_n(t_j) \right\} . \tag{29}$$

Therefore, we have

$$\lim_{N\to\infty} \left\{ \prod_{j=0}^{N-1} \langle \psi_n(t_j) \mid \psi_n(t_{j+1}) \rangle \right\} \to \exp\Big\{ -i\gamma_n(T) \Big\}, \tag{30}$$

in the limit that $N \to \infty$. This is analogous to the so-called Wilson loop in field theories.

In this formulation, it is also simple to generalize to the case that the energy $E_n(t)$ and $E_{n+1}(t)$ is doubly dengerate at some time $t = t_0$. It is obvious that the adiabatic approximation breaks down for these two states. In such a case, we cannot consider the two states separately. We need to calculate the following two by two matrix instead:

$$\delta \boldsymbol{W}_j = \begin{pmatrix} \langle \psi_n(t_j) \mid \psi_n(t_{j+1}) \rangle & \langle \psi_n(t_j) \mid \psi_{n+1}(t_{j+1}) \rangle \\ \langle \psi_{n+1}(t_j) \mid \psi_n(t_{j+1}) \rangle & \langle \psi_{n+1}(t_j) \mid \psi_{n+1}(t_{j+1}) \rangle \end{pmatrix} . \tag{31}$$

To find the geometric phases, we need to calculate the product first:

$$\boldsymbol{W} = \delta \boldsymbol{W}_0 \delta \boldsymbol{W}_1 \ldots \delta \boldsymbol{W}_{N-1}. \tag{32}$$

Finally, we diagonalize the matrix $\boldsymbol{M}$ to obatin two phases. In the limit that $N \to \infty$, this will give rsie to the non-abelian Wilson loop. Since matrices product is not commutative in general, path-ordering (time-ordering) is required in doing such a calculation in contrast to the abelian case.

We can relate the Berry phase to the so-called Zak phase, if change t to k and T to 2π:

$$\gamma_n = i \int_0^{2\pi} dt \langle \psi_n(k) \mid \partial_t \psi_n(k) \rangle . \tag{33}$$

For the SSH model, $\gamma_n = 0$ if the system is in the trivial phase, while $\gamma_n = \pi$ if the system is in the topological phase. In general, the Zak phase is related to the winding number through the following identity:

$$\gamma_n = \nu_n \pi. \tag{34}$$

Similarly, we will see that the non-abelian Wilson loop is related to the generalized "Zak phase", when we consider systems with four or more bands.

3. Adiabatic Charge-Pumping of the RM Model

Let's now consider the RM model by with time-dependent hopping amplitudes and chemical potential

$$\boldsymbol{H}(t)=\sum_{j=-\infty}^{\infty}\Big\{u(t)\boldsymbol{a}_j^\dagger\boldsymbol{a}_j-u(t)\boldsymbol{b}_j^\dagger\boldsymbol{b}_j-v(t)\boldsymbol{b}_j^\dagger\boldsymbol{a}_j-w(t)\boldsymbol{a}_{j+1}^\dagger\boldsymbol{b}_j-v(t)\boldsymbol{a}_j^\dagger\boldsymbol{b}_j-w(t)\boldsymbol{b}_j^\dagger\boldsymbol{a}_{j+1}\Big\}. \quad (35)$$

We may express the above Hamiltonian in the momentum space:

$$\boldsymbol{H}(t)=\int_0^{2\pi}\frac{dp}{2\pi}\left(\boldsymbol{a}^\dagger(p),\boldsymbol{b}^\dagger(p)\right)\mathcal{H}(t,p)\begin{pmatrix}\boldsymbol{a}(p)\\ \boldsymbol{b}(p)\end{pmatrix}, \quad (36)$$

where the Bloch Hamiltonian $\mathcal{H}$ is given by

$$\mathcal{H}(t,p)=\begin{pmatrix}u(t) & \bar{h}(t,p)\\ h(t,p) & -u(t)\end{pmatrix}, \quad (37)$$

with $h(t,p)=-\big\{v(t)+w(t)e^{ip}\big\}$. Equivalently, we have

$$\mathcal{H}(t,p)=h_1(t,p)\sigma_1+h_2(t,p)\sigma_2+h_3(t,p)\sigma_3, \quad (38)$$

with $h_1(t,p)=v(t)+w(t)\cos p, h_2(t,p)=w(t)\sin p, h_3(t,p)=u(t)$.

The instantaneous energy eigenstates of $\mathcal{H}(t,p)$ satisfy the following eigen equation

$$\mathcal{H}(t,p)\mid\psi_\pm(p)\rangle=\pm\omega(t)\mid\psi_\pm(t,p)\rangle. \quad (39)$$

Here, $\omega(t)=\sqrt{u(t)^2+v(t)^2+w(t)^2+2v(t)w(t)\cos(p)}$. They are given by:

$$\mid\psi_\pm(t,p)\rangle=\frac{1}{\sqrt{2\omega(t)\big\{\omega(t)\mp u(t)\big\}}}\begin{pmatrix}\pm\bar{h}(t,p)\\ \omega\mp u(t)\end{pmatrix}. \quad (40)$$

To be specific, let consider

$$u(t)=\sin(2\pi t/T),\ v(t)=v_0+\cos(2\pi t/T),\ w(t)=1, \quad (41)$$

so that

$$h_1(t,p)=v_0+\cos(2\pi t/T)+\cos p, h_2(t,p)=\sin p, h_3(t,p)=\sin(2\pi t/T). \quad (42)$$

As a result, $\boldsymbol{H}(t,p)$ and $\mathcal{H}(t,p)$ are periodic with period T. Note that when $t=k\pi$, with k an integer $u(t)=0$. Hence, the model reduces to the SSH model at these instants. In particular,

$$\begin{cases}u(0)=0,\ v(0)=v_0+1,\ w(0)=1,\\ u(T/2)=0,\ v(T/2)=v_0-1,\ w(T/2)=1.\end{cases}$$

Therefore, if $v_0<2$, the SSH models at the instants $t=0$ and $t=T/2$ would be in the trivial and topological phases, repectively. In other words:

$$\begin{cases}\langle\boldsymbol{x}\rangle=0, \text{for } t=0;\\ \langle\boldsymbol{x}\rangle=\frac{1}{2}, \text{for } t=T/2.\end{cases}$$

Therefore, it seems that the average position of the charge is pushing to the right by one unit each cycle.

On the other hand, we may carry out a rotaion in the internal space so that

$$\tilde{\mathcal{H}}(t,p)=\mathbb{U}\mathcal{H}(t,p)\mathbb{U}^\dagger=h_1(t,p)\sigma_1+h_3(t,p)\sigma_2-h_2(t,p)\sigma_3. \quad (43)$$

For $p=0$ and $p=\pi$, we have

$$\begin{cases}\tilde{\mathcal{H}}(t,0)=\{v_0+1+\cos(2\pi t/T)\}\,\sigma_1+\sin(2\pi t/T)\sigma_2;\\ \tilde{\mathcal{H}}(t,\pi)=\{v_0-1+\cos(2\pi t/T)\}\,\sigma_1+\sin(2\pi t/T)\sigma_2.\end{cases} \quad (44)$$

This is again similar to the SSH model if we view $2\pi t/T$ as a momentum. When $v_0<2$, $\tilde{\mathcal{H}}(t,0)$ and $\tilde{\mathcal{H}}(t,\pi)$ would also be in the trivial and topological phases, repectively.

4. Qi-Wu-Zhang (QWZ) model, 2D Chern insulators

A 2D Chern insulators is described by the following Hamiltonian:

$$\begin{aligned}\boldsymbol{H} = \sum_{j_1,j_2}\Big\{ & \mu \boldsymbol{a}^\dagger_{j_1,j_2}\boldsymbol{a}_{j_1,j_2} - \mu \boldsymbol{b}^\dagger_{j_1,j_2}\boldsymbol{b}_{j_1,j_2} \\ & -\frac{1}{2}\boldsymbol{a}^\dagger_{j_1+1,j_2}\boldsymbol{a}_{j_1,j_2} - \frac{1}{2}\boldsymbol{a}^\dagger_{j_1,j_2}\boldsymbol{a}_{j_1+1,j_2} - \frac{1}{2}\boldsymbol{a}^\dagger_{j_1,j_2+1}\boldsymbol{a}_{j_1,j_2} - \frac{1}{2}\boldsymbol{a}^\dagger_{j_1,j_2}\boldsymbol{a}_{j_1,j_2+1} \\ & +\frac{1}{2}\boldsymbol{b}^\dagger_{j_1+1,j_2}\boldsymbol{b}_{j_1,j_2} + \frac{1}{2}\boldsymbol{b}^\dagger_{j_1,j_2}\boldsymbol{b}_{j_1+1,j_2} + \frac{1}{2}\boldsymbol{b}^\dagger_{j_1,j_2+1}\boldsymbol{b}_{j_1,j_2} + \frac{1}{2}\boldsymbol{b}^\dagger_{j_1,j_2}\boldsymbol{b}_{j_1,j_2+1} \\ & -\frac{i}{2}\boldsymbol{b}^\dagger_{j_1+1,j_2}\boldsymbol{a}_{j_1,j_2} + \frac{i}{2}\boldsymbol{a}^\dagger_{j_1,j_2}\boldsymbol{b}_{j_1+1,j_2} - \frac{i}{2}\boldsymbol{a}^\dagger_{j_1+1,j_2}\boldsymbol{b}_{j_1,j_2} + \frac{i}{2}\boldsymbol{b}^\dagger_{j_1,j_2}\boldsymbol{a}_{j_1+1,j_2} \\ & +\frac{1}{2}\boldsymbol{b}^\dagger_{j_1,j_2+1}\boldsymbol{a}_{j_1,j_2} + \frac{1}{2}\boldsymbol{a}^\dagger_{j_1,j_2}\boldsymbol{b}_{j_1,j_2+1} - \frac{1}{2}\boldsymbol{a}^\dagger_{j_1,j_2+1}\boldsymbol{b}_{j_1,j_2} - \frac{1}{2}\boldsymbol{b}^\dagger_{j_1,j_2}\boldsymbol{a}_{j_1,j_2+1}\Big\}. \end{aligned} \tag{45}$$

The Bloch Hamiltonian in the momentum space is given by

$$\mathcal{H}(\boldsymbol{p}) = h_1(\boldsymbol{p})\sigma_1 + h_2(\boldsymbol{p})\sigma_2 + h_3(\boldsymbol{p})\sigma_3 = \vec{h}(\boldsymbol{p})\cdot\vec{\sigma}. \tag{46}$$

Here, $\boldsymbol{p} = p_1\hat{\boldsymbol{p}}_1 + p_2\hat{\boldsymbol{p}}_2$, $h_1(\boldsymbol{p}) = -\sin p_1$, $h_2(\boldsymbol{p}) = -\sin p_2$, and $h_3(\boldsymbol{p}) = \mu - \cos p_1 - \cos p_2$. As $\boldsymbol{p}$ ranges over the 2D BZ: $(-\pi,\pi]\times(-\pi,\pi]$, the vector $\vec{h}(\boldsymbol{p})$ traces out a 2D surface in a 3D space.

$= \boldsymbol{r}(u,v)$ For a general surface in 3D, we can use two parameters to describe the surface:

$$\boldsymbol{r}(u,v) = \sum_{i=1}^{3} x_i(u,v)\hat{\boldsymbol{e}}_i, \quad \hat{\boldsymbol{e}}_i : \text{Cartesian basis}$$

By varying u and v, respectively, we have

$$\begin{cases}\boldsymbol{r}(u+du,v) - \boldsymbol{r}(u,v) = (\partial_u \boldsymbol{r})du, \\ \boldsymbol{r}(u,v+dv) - \boldsymbol{r}(u,v) = (\partial_v \boldsymbol{r})dv.\end{cases}$$

Hence, the small parallelogram spanned by the varying u and v is given by

$$d\boldsymbol{S} = \{(\partial_u\boldsymbol{r})du\}\times\{(\partial_v\boldsymbol{r})dv\} = (\partial_u\boldsymbol{r}\times\partial_v\boldsymbol{r})du\,dv = dS\,\hat{\boldsymbol{n}}$$

The magnitude dS gives us the area of the parallelogram and the direction $\hat{\boldsymbol{n}}$ points along the normal of the parallelogram. For example, let's consider the ellipsoid:

$$\frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} = 1.$$

Let $x = a\sin\theta\cos\phi$, $\quad y = a\sin\theta\sin\phi$, $\quad z = b\cos\theta$, where $(u,v) = (\theta,\phi)$, $\quad \theta\in[0,\pi]$, and $\phi\in[0,2\pi)$.

As a result, we can define the 2D winding number (Chern number) by:

$$C = \frac{1}{4\pi}\int_{\boldsymbol{p}} d^2\boldsymbol{p}\frac{\vec{h}(\boldsymbol{p})\cdot\Big\{\partial_1\vec{h}(\boldsymbol{p})\times\partial_2\vec{h}(\boldsymbol{p})\Big\}}{|\vec{h}(\boldsymbol{p})|^3}. \tag{47}$$

Note that the 4π here, is the solid angle and $\partial_i \equiv \frac{\partial}{\partial p_i}$. On the other hand, we can define the ficticoius vector potential:

$$A_i(\boldsymbol{p}) = i\,\langle\,\psi(\boldsymbol{p})\,|\,\partial_i\psi(\boldsymbol{p})\,\rangle\,. \tag{48}$$

Here, $|\psi(\boldsymbol{p})\rangle$ is the energyeigen state, which satisfies

$$\mathcal{H}(\boldsymbol{p})|\psi(\boldsymbol{p})\rangle = E(\boldsymbol{p})|\psi(\boldsymbol{p})\rangle\,. \tag{49}$$

Hence, the fictitious magnetic field and flux are given by

$$B(\boldsymbol{p}) = \nabla \times \boldsymbol{A}(\boldsymbol{p}) = \partial_1 A_2(\boldsymbol{p}) - \partial_2 A_1(\boldsymbol{p}), \tag{50}$$

and

$$\Phi = \int_{\boldsymbol{p}} d^2\boldsymbol{p} B(\boldsymbol{p}), \tag{51}$$

respectively. The energy eigenstates in the upper and lower bands satisfy

$$\mathcal{H}(\boldsymbol{p})|\psi_\pm(\boldsymbol{p})\rangle = \pm\omega(\boldsymbol{p})|\psi_\pm(\boldsymbol{p})\rangle\,, \tag{52}$$

respectively, with $\omega(\boldsymbol{p}) = |\vec{h}(\boldsymbol{p})|$. Assumiming $|\psi_\pm(\boldsymbol{p})\rangle = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$, we have

$$\begin{pmatrix}\mu \mp \omega(\boldsymbol{p}) & \bar{h}(\boldsymbol{p})\\ h(\boldsymbol{p}) & -\mu \mp \omega(\boldsymbol{p})\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = 0. \tag{53}$$

If we use the first row of the equation to solve for α and β, it is not difficult to see that

$$|\psi_\pm(\boldsymbol{p})\rangle_{\mathrm{I}} = \mathcal{N}_{\mathrm{I},\pm}\begin{pmatrix}\pm\bar{h}(\boldsymbol{p})\\ \omega(\boldsymbol{p}) \mp h_3(\boldsymbol{p})\end{pmatrix}, \tag{54}$$

with $\mathcal{N}_{\mathrm{I},\pm}^{-1} = \sqrt{2\omega(\boldsymbol{p})\Big\{\omega(\boldsymbol{p}) \mp h_3(\boldsymbol{p})\Big\}}$. The subscript I here denotes that "gauge" I is chosen. On the other, if we use the second row of the equation to solve for α and β, it is not difficult to see that

$$|\psi_\pm(\boldsymbol{p})\rangle_{\mathrm{II}} = \mathcal{N}_{\mathrm{II},\pm}\begin{pmatrix}\omega(\boldsymbol{p}) \pm h_3(\boldsymbol{p})\\ \pm h(\boldsymbol{p})\end{pmatrix}, \tag{55}$$

with $\mathcal{N}_{\mathrm{II},\pm}^{-1} = \sqrt{2\omega(\boldsymbol{p})\big\{\omega(\boldsymbol{p}) \pm h_3(\boldsymbol{p})\big\}}$.

If we consider the half-filled case, then we will focus on the energy eigenstate in the lower band, i. e. $|\psi_-(\boldsymbol{p})\rangle_{\mathrm{I}}$ and $|\psi_-(\boldsymbol{p})\rangle_{\mathrm{II}}$. Since

$$\begin{cases}\mathcal{N}_{\mathrm{I},-}^{-1} = \sqrt{2\omega(\boldsymbol{p})\big\{\omega(\boldsymbol{p}) + h_3(\boldsymbol{p})\big\}};\\ \mathcal{N}_{\mathrm{II},-}^{-1} = \sqrt{2\omega(\boldsymbol{p})\big\{\omega(\boldsymbol{p}) - h_3(\boldsymbol{p})\big\}},\end{cases}$$

the wave functions in gauge I and II are ill-defined when $\omega(\boldsymbol{p}) + h_3(\boldsymbol{p}) = 0$ and $\omega(\boldsymbol{p}) - h_3(\boldsymbol{p})$, respectively. As $\omega(\boldsymbol{p}) = \sqrt{h_1^2(\boldsymbol{p}) + h_2^2(\boldsymbol{p}) + h_3^2(\boldsymbol{p})}$, this can happen only if $h_1(\boldsymbol{p}) = h_2(\boldsymbol{p}) = 0$. Thus, we must have $\boldsymbol{p}$ to be one of the time-reversal-invariant-momentua(TRIM):
$\boldsymbol{p}_A = (0,0)$, $\boldsymbol{p}_B = (\pi,0)$, $\boldsymbol{p}_C = (0,\pi)$, $\boldsymbol{p}_D = (\pi,\pi)$.
At these momenta, $h_3(\boldsymbol{p}_A) = \mu - 2$, $h_3(\boldsymbol{p}_D) = \mu + 2$, and $h_3(\boldsymbol{p}_B) = h_3(\boldsymbol{p}_C) = \mu$,
and $\omega(\boldsymbol{p}) = |h_3(\boldsymbol{p})|$. Thus,

$$\begin{cases}|\psi_-(\boldsymbol{p})\rangle_{\mathrm{I}} & \text{is ill}-\text{defined if } h_3(\boldsymbol{p}) < 0;\\ |\psi_-(\boldsymbol{p})\rangle_{\mathrm{II}} & \text{is ill}-\text{defined if } h_3(\boldsymbol{p}) > 0.\end{cases}$$

In sum,

$$\begin{cases} \mu > 2, & |\psi_-(\boldsymbol{p})\rangle_{\rm I} \text{ is always well}-\text{defined and } |\psi_-(\boldsymbol{p})\rangle_{\rm II} \text{is always ill}-\text{defined.} \\ 0 < \mu < 2, & |\psi_-(\boldsymbol{p})\rangle_{\rm I} \text{ is ill}-\text{defined at } \boldsymbol{p}_A \text{ and is well}-\text{defined at } \boldsymbol{p}_B, \boldsymbol{p}_C, \boldsymbol{p}_D; \\ & |\psi_-(\boldsymbol{p})\rangle_{\rm II} \text{ is well}-\text{defined at } \boldsymbol{p}_A \text{ and is ill}-\text{defined at } \boldsymbol{p}_B, \boldsymbol{p}_C, \boldsymbol{p}_D. \\ -2 < \mu < 0, & |\psi_-(\boldsymbol{p})\rangle_{\rm I} \text{ is ill}-\text{defined at } \boldsymbol{p}_A, \boldsymbol{p}_B, \boldsymbol{p}_C \text{ and is well}-\text{defined at } \boldsymbol{p}_D; \\ & |\psi_-(\boldsymbol{p})\rangle_{\rm II} \text{ is well}-\text{defined at } \boldsymbol{p}_A, \boldsymbol{p}_B, \boldsymbol{p}_C \text{ and is ill}-\text{defined at } \boldsymbol{p}_D. \\ \mu < -2, & |\psi_-(\boldsymbol{p})\rangle_{\rm I} \text{ is always ill}-\text{defined and } |\psi_-(\boldsymbol{p})\rangle_{\rm II} \text{ is always well}-\text{defined.} \end{cases}$$

As we shall see, this translates to

$$\begin{cases} |\mu| > 2, & \text{the system is in the trivial phase.} \\ |\mu| < 2, & \text{the system is in the topological phase.} \end{cases}$$

$$|\psi_-(\boldsymbol{p})\rangle_I = N_{\rm I,-}(\boldsymbol{p}) \begin{pmatrix} -\bar{h}(\boldsymbol{p}) \\ h_3(\boldsymbol{p}) + \omega(\boldsymbol{p}) \end{pmatrix}, \quad N_{\rm I,-}(\boldsymbol{p}) = \{2\omega(\omega + h_3)\}^{-1/2}$$

$$|\psi_-(\boldsymbol{p})\rangle_{\rm II} = N_{\rm II,-}(\boldsymbol{p}) \begin{pmatrix} \omega(\boldsymbol{p}) - h_3(\boldsymbol{p}) \\ -h(\boldsymbol{p}) \end{pmatrix}, \quad N_{\rm II,-}(\boldsymbol{p}) = \{2\omega(\omega - h_3)\}^{-1/2}$$

Gauge transformation:

$$|\psi_-(\boldsymbol{p})\rangle_{\rm II} = -e^{i\phi_{\boldsymbol{p}}} |\psi_-(\boldsymbol{p})\rangle_I, \quad \phi_{\boldsymbol{p}} = \tan^{-1}\left\{\frac{h_2(\boldsymbol{p})}{h_1(\boldsymbol{p})}\right\}$$

$$\begin{cases} A_1^{\rm II} = i\,{}_{\rm II}\langle\psi_-|\partial_1\psi_-\rangle_{\rm II} = A_1^{\rm I} - \partial_1\phi_{\boldsymbol{p}}, \\ A_2^{\rm II} = i\,{}_{\rm II}\langle\psi_-|\partial_2\psi_-\rangle_{\rm II} = A_2^{\rm I} - \partial_2\phi_{\boldsymbol{p}}, \end{cases} \quad \Rightarrow \boldsymbol{A}^{\rm II} = \boldsymbol{A}^{\rm I} - \boldsymbol{\nabla}\phi_{\boldsymbol{p}}. \tag{56}$$

$$\boldsymbol{B}^{\rm II} = \boldsymbol{\nabla} \times \boldsymbol{A}^{\rm II} = \boldsymbol{\nabla} \times \boldsymbol{A}^{\rm I} = \boldsymbol{B}^{\rm I}, \tag{57}$$

if $\phi_{\boldsymbol{p}}$ does not have a singularity.

Homework:

(a) Calculate the surface area of the ellipsoid.

(b) Consider the ellipsoid and $\boldsymbol{E} = \frac{\boldsymbol{r}}{|\boldsymbol{r}|^3}$, where $r^2 = x^2 + y^2 + z^2$. Calculate $\int(\boldsymbol{E} \cdot d\boldsymbol{S})$.

Homework: Show that $\frac{\Phi}{2\pi}$ and C are identical.

Making use of Stokes' theorem,

$$\Phi = \int_S d^2\boldsymbol{p} B(\boldsymbol{p}) = \int_{\partial S} \boldsymbol{A}(\boldsymbol{p}) \cdot d\boldsymbol{p}. \tag{58}$$

Here, S and ∂S is its boundary. Since S is the BZ, which is a torus, it has no boundary. If $\boldsymbol{A}(\boldsymbol{p})$ is well-defined over the whole BZ, then Φ would vanishing. Thus, when $|\mu| > 2$, we can choose the propper gauge so that $\boldsymbol{A}(\boldsymbol{p})$ is always well-defined. As a result, the Chern number would be zero and the system is in the trivial phase.

When $|\mu| < 2$, no matter which gauge we choose, we cannot make $\boldsymbol{A}(\boldsymbol{p})$ well-defined globally. To be specific, let's consider the case that $0 < \mu < 2$. As $|\psi_-(\boldsymbol{p})\rangle_{\rm I}$ is ill-defined at $\boldsymbol{p}_A$, we must choose gauge II around the neighborbood of $\boldsymbol{p}_A$ and gauge I outside the region. If we make the

neighbor around $\boldsymbol{p}_A$ small enough, we can do expansion near $\boldsymbol{p} = \boldsymbol{p}_A$: $p_1 = \epsilon\delta p_1, p_2 = \epsilon\delta p_2$, with $\epsilon \ll 1$ a parameter for booking. Now, let's divide the BZ into two regions. $|\psi_-(\boldsymbol{p})\rangle_{\rm I}$ and $|\psi_-(\boldsymbol{p})\rangle_{\rm II}$ are well-defined in regions I and II, respectively, with region II being a small disc around $\boldsymbol{p}_A$. As a result,

$$\Phi = \int_{S_{\rm I}} d^2\boldsymbol{p} B_{\rm I}(\boldsymbol{p}) + \int_{S_{\rm II}} d^2\boldsymbol{p} B_{\rm II}(\boldsymbol{p}) \tag{59}$$

Making use of Stokes' theorem,

$$\Phi = \int_{\partial S_{\rm I}} \boldsymbol{A}_{\rm I}(\boldsymbol{p}) \cdot d\boldsymbol{p} + \int_{\partial S_{\rm II}} \boldsymbol{A}_{\rm II}(\boldsymbol{p}) \cdot d\boldsymbol{p}. \tag{60}$$

Since regions I and II are on the opposite sides of the boundary, we have $\partial S_{\rm I} = -\partial S_{\rm II}$. Therefore,

$$\Phi = \int_{\partial S_{\rm II}} \Big\{ \boldsymbol{A}_{\rm II}(\boldsymbol{p}) - \boldsymbol{A}_{\rm I}(\boldsymbol{p}) \Big\} \cdot d\boldsymbol{p} = -\oint \boldsymbol{\nabla}\phi_{\boldsymbol{p}} \cdot d\boldsymbol{p} = -\oint d\phi_{\boldsymbol{p}} = -2\pi. \tag{61}$$

Thus, the Chern number $C = -1$ for the case that $0 < \mu < 2$. Similarly, one can show that the Chern number $C = 1$ for the case that $-2 < \mu < 0$. This cofirm that the system is in the topological phase when $|\mu| < 2$.

We can also do the cauculation explicitly. In the neiborhood around, $\boldsymbol{p} = \boldsymbol{p}_A$, we have $h_1(\boldsymbol{p}) \approx -\epsilon\delta p_1$, $h_2(\boldsymbol{p}) \approx -\epsilon\delta p_2$, and $h_3(\boldsymbol{p}) \approx \mu - 2 + \frac{\epsilon^2}{2}|\delta\boldsymbol{p}|^2$. Thus, $\omega(\boldsymbol{p}) \approx (2-\mu) + \frac{(\mu-1)}{2(2-\mu)}\epsilon^2|\delta\boldsymbol{p}|^2$. and $\omega(\boldsymbol{p}) + h_3(\boldsymbol{p}) \approx \frac{1}{2(2-\mu)}\epsilon^2|\delta\boldsymbol{p}|^2$. $\mathcal{N}_{\rm I,-}^{-1} \approx \epsilon|\delta\boldsymbol{p}|$. It is then not difficult to see that

$$|\psi_-(\boldsymbol{p})\rangle_{\rm I} \approx \begin{pmatrix} -e^{-i\phi_{\boldsymbol{p}}} \\ 0 \end{pmatrix}, \quad {}_{\rm I}\langle\,\psi_-(\boldsymbol{p})| \approx \begin{pmatrix} -e^{i\phi_{\boldsymbol{p}}} & 0 \end{pmatrix}, \tag{62}$$

with $\delta p_1 + i\delta p_2 = |\delta\boldsymbol{p}|e^{i\phi_{\boldsymbol{p}}}$.

$$|\boldsymbol{\nabla}\psi_-(\boldsymbol{p})\rangle_{\rm I} \approx \begin{pmatrix} i\big\{\boldsymbol{\nabla}\phi_{\boldsymbol{p}}\big\}e^{-i\phi_{\boldsymbol{p}}} \\ 0 \end{pmatrix},$$

$$\boldsymbol{A}_{\rm I}(\boldsymbol{p}) = i\,{}_{\rm I}\big\langle\psi_-(\boldsymbol{p})\big|\boldsymbol{\nabla}\psi_-(\boldsymbol{p})\big\rangle_{\rm I} = \boldsymbol{\nabla}\phi_{\boldsymbol{p}}. \tag{63}$$

$$\oint \boldsymbol{A}_{\rm I}(\boldsymbol{p}) \cdot d\boldsymbol{p} = \oint \boldsymbol{\nabla}\phi_{\boldsymbol{p}} \cdot d\boldsymbol{p} = \oint d\phi_{\boldsymbol{p}} = -2\pi. \tag{64}$$

This is consistent with our previous result in eq. (61). Note that in the above calculation, we ignore the contribution from region II completely. This is reasonable because region II is a just small neighborhood around $\boldsymbol{p}_A$ and both $\boldsymbol{A}_{\rm II}(\boldsymbol{p})$ and $B_{\rm II}(\boldsymbol{p})$ are regular there.

Homework: Carry out similar analysis for th ecase that $-2 < \mu < 0$.

Analogy with Electrodynamics:

- 3D point charge: $\boldsymbol{E} = \frac{q\boldsymbol{r}}{4\pi\epsilon_0 r^3}$, $V = \frac{q}{4\pi\epsilon_0 r} \Rightarrow \boldsymbol{\nabla}\cdot\boldsymbol{E} = \frac{q}{\epsilon_0}\delta^{(3)}(\boldsymbol{r})$.
- 2D point charge (3D line charge): $\boldsymbol{E} = \frac{\lambda\boldsymbol{r}}{2\pi\epsilon_0 r^2}$, $V = -\frac{\lambda}{2\pi\epsilon_0}\ln r \Rightarrow \boldsymbol{\nabla}\cdot\boldsymbol{E} = \frac{\lambda}{\epsilon_0}\delta^{(2)}(\boldsymbol{r})$.

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = \partial_x E_x + \partial_y E_y, \quad E_x = \frac{\cos\phi}{r}, \quad E_y = \frac{\sin\phi}{r}.$$

$$A_x^{\rm I} \propto \partial_x\phi = \partial_x\left(\tan^{-1}\frac{y}{x}\right) = \frac{-\frac{y}{x^2}}{1+(\frac{y}{x})^2} = \frac{-y}{x^2+y^2} = -\frac{\sin\phi}{r} \propto -E_y;$$

$$A_y^{\rm I} \propto \partial_y\phi = \frac{\frac{1}{x}}{1+(\frac{y}{x})^2} = \frac{x}{x^2+y^2} = \frac{\cos\phi}{r} \propto E_x.$$

$$B^{\rm I} = \partial_x A_y^{\rm I} - \partial_y A_x^{\rm I} = \partial_x E_x - \partial_y(-E_y) = \vec{\nabla}\cdot\vec{E} \propto \delta^{(2)}(\boldsymbol{r}).$$

Therefore, a infinitely thin solenoid is analogous to a point charge in 2D mathematically.

Relation between Chern number and 1D winding number:

$h_3 = \mu - (\cos p_1 + \cos p_2), \quad h_1 = -\sin p_1, \quad h_2 = -\sin p_2.$

(i) $p_2 = 0 \Rightarrow h_3 = (\mu - 1) - \cos p_1$, $h_1 = -\sin p_1$, $h_2 = 0$:

$H(p_2 = 0) = \{(\mu - 1) - \cos p_1\}\sigma_3 - \{\sin p_1\}\sigma_1.$

It is known that

$$H_{\rm SSH} = -(t_0 + t_1 \cos p)\sigma_1 - t_1 \sin p\, \sigma_2. \tag{65}$$

By unitary transformation $\mathbb{U}\sigma_3\mathbb{U}^\dagger = \sigma_1$, $\mathbb{U}\sigma_1\mathbb{U}^\dagger = \sigma_2$, and $\mathbb{U}\sigma_2\mathbb{U}^\dagger = \sigma_3$.
we see that $H(p_2 = 0) \simeq H_{\rm SSH}$, with $t_0 = \mu - 1$ and $t_1 = 1$.
$\nu(0) = 1$ for $|\mu - 1| < 1$; $\nu(0) = 0$ for $|\mu - 1| > 1$.

(ii) $p_2 = \pi \Rightarrow h_3 = (\mu + 1) - \cos p_1$, $h_1 = -\sin p_1$, $h_2 = 0$:

$H(p_2 = \pi) = \{(\mu + 1) - \cos p_1\}\sigma_3 - \{\sin p_1\}\sigma_1.$

Hence, $H(p_2 = 0) \simeq H_{\rm SSH}$, with $t_0 = \mu + 1$ and $t_1 = 1$.
$\nu(\pi) = 1$ for $|\mu + 1| < 1$; $\nu(0) = 0$ for $|\mu + 1| > 1$.

Let's consider the "magnetic flux" contributed from the upper BZ:

$$\int_{\Omega_{\rm u}} d^2\boldsymbol{p}\, B = \int_0^\pi dp_2 \left(\int_{-\pi}^{\pi} B\, dp_1 \right) = \int_0^\pi \int_{-\pi}^{\pi} (\boldsymbol{\nabla} \times \boldsymbol{A})\, dp_1 dp_2, \tag{66}$$

with $\Omega_{\rm u} = (-\pi, \pi] \times [0, \pi]$.

$$C/2 = \frac{1}{2\pi} \int_{p_2=0} (\boldsymbol{A} \cdot d\boldsymbol{p}) - \frac{1}{2\pi} \int_{p_2=\pi} (\boldsymbol{A} \cdot d\boldsymbol{p}) = \frac{1}{2}\{\nu(0) - \nu(\pi)\}. \tag{67}$$

Here, we have made use of the fact that B is an even function of p_1 and p_2.

What if we consider the "magnetic flux" contributed from the lower BZ?

$$\int_{\Omega_{\rm l}} d^2\boldsymbol{p}\, B = \int_{-\pi}^0 dp_2 \left(\int_{-\pi}^{\pi} B\, dp_1 \right) = \int_{-\pi}^0 \int_{-\pi}^{\pi} (\boldsymbol{\nabla} \times \boldsymbol{A})\, dp_1 dp_2, \tag{68}$$

with $\Omega_{\rm l} = (-\pi, \pi] \times [-\pi, 0]$.

$$C/2 = \frac{1}{2\pi} \int_{p_2=-\pi} (\boldsymbol{A} \cdot d\boldsymbol{p}) - \frac{1}{2\pi} \int_{p_2=0} (\boldsymbol{A} \cdot d\boldsymbol{p}) = \frac{1}{2}\{\nu(-\pi) - \nu(0)\}. \tag{69}$$

Since we expect that $\nu(-\pi) = \nu(\pi)$, this seems to contract the result from eq. (67). The subtlety arises from the fact that after the rotation in the internal space, we have
$\tilde{h}_1 = h_3 = \mu - (\cos p_1 + \cos p_2)$, $\tilde{h}_2 = h_1 = -\sin p_1$, and $\tilde{h}_3 = h_2 = -\sin p_2$.
It is obvious that $\tilde{h}_3$ is negative for $0 < p_2 < \pi$ and positive for $-\pi < p_2 < 0$. Hence, $\big|\tilde{\psi}_-(\boldsymbol{p})\big\rangle_{\rm I}$ is well-defined in the lower patch of the BZ, while $\big|\tilde{\psi}_-(\boldsymbol{p})\big\rangle_{\rm II}$ is well-defined in the upper patch of the BZ. Therefore, eqs. (67) and (69) should be modified to

$$C/2 = \frac{1}{2\pi} \int_{p_2=0} (\boldsymbol{A}_{\rm II} \cdot d\boldsymbol{p}) - \frac{1}{2\pi} \int_{p_2=\pi} (\boldsymbol{A}_{\rm II} \cdot d\boldsymbol{p}) = \frac{1}{2}\{\nu_{\rm II}(0) - \nu_{\rm II}(\pi)\}; \tag{70}$$

$$C/2 = \frac{1}{2\pi} \int_{p_2=\pi} (\boldsymbol{A}_{\rm I} \cdot d\boldsymbol{p}) - \frac{1}{2\pi} \int_{p_2=0} (\boldsymbol{A}_{\rm I} \cdot d\boldsymbol{p}) = \frac{1}{2}\{\nu_{\rm I}(\pi) - \nu_{\rm I}(0)\}. \tag{71}$$

Then, there would be no contradiction between them.

Similar to the QWZ model before the rotation in the internal space, the sigularities of the wave function $\left|\tilde{\psi}_-(\boldsymbol{p})\right\rangle_{\rm I}$ and $\left|\tilde{\psi}_-(\boldsymbol{p})\right\rangle_{\rm II}$ would appear when $\tilde{h}_1 = 0$ and $\tilde{h}_2 = 0$, i. e.

$$p_1 = 0, \text{and } \mu - 1 - \cos p_2 = 0, \tag{72}$$

$$p_1 = \pi, \text{and } \mu + 1 - \cos p_2 = 0. \tag{73}$$

Let's use the cases that $\mu > 2$ and $0 < \mu < 2$ to check the results in eqs. (70) and (71) .

(i) $\mu > 2$:
Neither of eqs. (72) and (73) have any solution. Since there is no singularity, we are free to choose either the upper or lower patches to do the integration. Since both $\mu - 1$ and $\mu + 1$ are larger than one, $\nu_{\rm I}(0) = \nu_{\rm II}(0) = \nu_{\rm I}(\pi) = \nu_{\rm II}(\pi) = 0$. Thus, it is obvious that both equations in (70) and (71) hold.

(ii) $0 < \mu < 2$:
Eq. (72) has solutions at $p_2 = -P_0, P_0$ with $P_0 = \cos^{-1}(\mu - 1)$, but eq. (73) does not have any solution. Therefore, $\left|\tilde{\psi}_-(\boldsymbol{p})\right\rangle_{\rm I}$ is singular at $\tilde{\boldsymbol{p}}_C = (0, P_0)$, while $\left|\tilde{\psi}_-(\boldsymbol{p})\right\rangle_{\rm II}$ is singular at $\tilde{\boldsymbol{p}}_A = (0, -P_0)$. In gauge I, we will integrate the magnetic field over the lower patch of the BZ to avoid the sigularity of $\left|\tilde{\psi}_-(\boldsymbol{p})\right\rangle_{\rm I}$. Thus, we can use eq. (71) to calculate C. Since $\mu - 1 < 1$ and $\mu + 1 > 1$, we have $\nu_{\rm I}(0) = 1$ and $\nu_{\rm I}(\pi) = 0$. Putting all these together, we achieve $C = -1$.

On the other hand, if we choose gauge II, then we need to integrate the magnetic field over the upper patch of the BZ to avoid the sigularity of $\left|\tilde{\psi}_-(\boldsymbol{p})\right\rangle_{\rm II}$. Thus, we must use eq. (70) to calculate C instead. Since $\mu - 1 < 1$ and $\mu + 1 > 1$, we have $\nu_{\rm II}(0) = -1$ and $\nu_{\rm II}(\pi) = 0$. Putting all these together, we again arrive at $C = -1$.

Discrete symmetries of the Newton equation and the Maxwell equations:

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho/\epsilon, \quad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$$

$$\vec{\nabla} \times \boldsymbol{B} - \frac{1}{\mu\epsilon}\partial_t \boldsymbol{E} = \mu \boldsymbol{j}, \quad \boldsymbol{\nabla} \times \boldsymbol{E} + \partial_t \boldsymbol{B} = 0$$

Parity (Space Inversion)/ Mirror Reflection:

$$\boldsymbol{r} \to -\boldsymbol{r}, \quad \boldsymbol{p} \to -\boldsymbol{p}, \quad \boldsymbol{\nabla} \to -\boldsymbol{\nabla}, \quad \boldsymbol{j} = \rho \boldsymbol{v} \to -\boldsymbol{j}$$

$\boldsymbol{p}$: vector, $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$: axial vector.
$\boldsymbol{F} = m\boldsymbol{a}$: parity invariant.
If Maxwell equations are invariant under parity:

$$\begin{cases} \boldsymbol{E} \to -\boldsymbol{E} & \text{(polar vector)} \\ \boldsymbol{B} \to \boldsymbol{B} & \text{(axial / pseudo vector)} \end{cases}$$

Time Reversal ($t \to -t$):

$$\boldsymbol{r} \to \boldsymbol{r}, \quad \boldsymbol{p} \to -\boldsymbol{p}, \quad \boldsymbol{j} \to -\boldsymbol{j}$$

$\boldsymbol{F} = m\boldsymbol{a}$: Time reversal invariant.
If Maxwell equations are invariant under time reversal: $\boldsymbol{E} \to \boldsymbol{E}$, $\boldsymbol{B} \to -\boldsymbol{B}$.

For the QWZ model, we may now define the Wilson loop that depends on p_2. To calculate it numerically, we again discretize p_1 by introducing a large even number $N = 2M$. Let's define

$\Delta = 2\pi/N$ and $\tilde{p}_j = j * \Delta$, with $j = -M, -M+1, \ldots, -1, 0, 1, \ldots, M-1$. Now, let's consider the following inner product:

$$\delta W(\tilde{p}_j, p_2) \equiv \langle\, \psi(\tilde{p}_j, p_2) \,|\, \psi(\tilde{p}_{j+1}, p_2) \,\rangle\,. \tag{74}$$

As a result, the Wilson loop is given by

$$\begin{aligned} W(p_2) &= \prod_{j=-M}^{M-1} \delta W(\tilde{p}_j, p_2), \\ W(p_2) &= \delta W(-\pi, p_2)\delta W(-\pi+\Delta, p_2)\ldots\delta W(0, p_2)\ldots\delta W(\pi-2\Delta, p_2)\delta W(\pi-\Delta, p_2). \end{aligned} \tag{75}$$

Obviously, in the limit that $N \to \infty$

$$W(p_2) = \exp\Big\{-i\int_{-\pi}^{\pi} dp_1 A_1(p_1, p_2)\Big\}. \tag{76}$$

Assume that $\boldsymbol{\pi}$ is the space inversion operator, so that $\boldsymbol{\pi p \pi} = -\boldsymbol{p}$. If the system is invariant under the space inversion, then we have

$$\begin{cases} \boldsymbol{\pi}\,|\,\psi(\tilde{p}_j, p_2)\,\rangle = e^{i\tilde{\phi}(\tilde{p}_j, p_2)}\,|\,\psi(\tilde{p}_{-j}, -p_2)\,\rangle\,, \\ \boldsymbol{\pi}\,|\,\psi(\tilde{p}_{-j}, -p_2)\,\rangle = e^{-i\tilde{\phi}(\tilde{p}_j, p_2)}\,|\,\psi(\tilde{p}_j, p_2)\,\rangle\,. \end{cases} \tag{77}$$

As a result,

$$\delta W(\tilde{p}_{-j-1}, p_2) = \langle\, \psi(\tilde{p}_{-j-1}, p_2) \,|\, \psi(\tilde{p}_{-j}, p_2) \,\rangle = e^{-i\left\{\tilde{\phi}(\tilde{p}_j, p_2) - \tilde{\phi}(\tilde{p}_{j+1}, p_2)\right\}}\left\{\delta W(\tilde{p}_j, -p_2)\right\}^*. \tag{78}$$

Since $p_2 = 0, \pi$ are invariant under space inversion, we see

$$\begin{aligned} \delta W(\tilde{p}_{-j-1}, 0) &= e^{-i\left\{\tilde{\phi}(\tilde{p}_j, 0) - \tilde{\phi}(\tilde{p}_{j+1}, 0)\right\}}\left\{\delta W(\tilde{p}_j, 0)\right\}^*, \\ \delta W(\tilde{p}_{-j-1}, \pi) &= e^{-i\left\{\tilde{\phi}(\tilde{p}_j, \pi) - \tilde{\phi}(\tilde{p}_{j+1}, \pi)\right\}}\left\{\delta W(\tilde{p}_j, \pi)\right\}^*. \end{aligned}$$

Consiquently,

$$\begin{aligned} \delta W(\tilde{p}_{-j-1}, 0)\delta W(\tilde{p}_j, 0) &= e^{-i\left\{\tilde{\phi}(\tilde{p}_j, 0) - \tilde{\phi}(\tilde{p}_{j+1}, 0)\right\}}\left|\delta W(\tilde{p}_j, 0)\right|^2 \approx e^{-i\left\{\tilde{\phi}(\tilde{p}_j, 0) - \tilde{\phi}(\tilde{p}_{j+1}, 0)\right\}}, \\ \delta W(\tilde{p}_{-j-1}, \pi)\delta W(\tilde{p}_j, \pi) &= e^{-i\left\{\tilde{\phi}(\tilde{p}_j, \pi) - \tilde{\phi}(\tilde{p}_{j+1}, \pi)\right\}}\left|\delta W(\tilde{p}_j, \pi)\right|^2 \approx e^{-i\left\{\tilde{\phi}(\tilde{p}_j, \pi) - \tilde{\phi}(\tilde{p}_{j+1}, \pi)\right\}}. \end{aligned}$$

In light of the structure of the above phase factors, most of them would cancel out except for those at the end points, i. e. $e^{i\tilde{\phi}(\boldsymbol{p}_A)}$, $e^{i\tilde{\phi}(\boldsymbol{p}_B)}$, $e^{i\tilde{\phi}(\boldsymbol{p}_C)}$ and $e^{i\tilde{\phi}(\boldsymbol{p}_D)}$. Since $\boldsymbol{p}_A, \boldsymbol{p}_B, \boldsymbol{p}_C$, and $\boldsymbol{p}_D$ are invariant under space inversion, all these phase factors would be either 1 or -1. Hence, $W(0)$ and $W(\pi)$ can only take on the values of 1 or -1. We can use the values of $W(0)$ and $W(\pi)$ to classify our system.

(i) $W(0)$ and $W(\pi)$ have the same value: the system is in the trivial phase.

(ii) $W(0)$ and $W(\pi)$ have opposite signs: the system is in the topological phase.

Homework: Use mathematica or python to write a program based on the Wilson loop formalism to check that the QWZ model is in the topopogical phase for $-2 < \mu < 2$.

Homework: Consider the four-band SSH model

$$\boldsymbol{H} = \sum_{j=-\infty}^{\infty}\Big\{-t_0\boldsymbol{b}_j^\dagger\boldsymbol{a}_j - t_1\boldsymbol{c}_j^\dagger\boldsymbol{b}_j - t_2\boldsymbol{d}_j^\dagger\boldsymbol{c}_j - t_3\boldsymbol{a}_{j+1}^\dagger\boldsymbol{d}_j + h.c.\Big\}. \tag{79}$$

(a) If the system has space inversion symmetry, what condition must the hopping parameters (t_0, t_1, t_2, t_3) satisfy?

(b) Use mathematica or python to write a program to check that the Wilson loop of each band of the four-band SSH model is quantized.

The Chern numbers of the following two QWZ models are opposite:
QWZ: $h_3 = \mu - (\cos p_1 + \cos p_2), \quad h_1 = -\sin p_1, \quad h_2 = -\sin p_2.$
QWZ′: $h_3' = \mu - (\cos p_1 + \cos p_2), \quad h_1' = -\sin p_1, \quad h_2' = \sin p_2.$

If we introduce next-to-nearest-neighbor hopping terms, then we have the so-called **Extended QWZ Model**:

$$h_3 = \mu - (\cos p_1 + \cos p_2) - t_2 \cos p_1 \cos p_2, \quad h_1 = -\sin p_1 \,(1 + t_2 \cos p_2)\,, \quad h_2 = -\sin p_2 \,(1 + t_2 \cos p_1)\,.$$

The possible values of the Chern number of the system are $\{-2, -1, 0, 1, 2\}$.
Homework: Consider the extended QWZ model mentined above.

(a) Follow the analysis carried out for the QWZ model and find all the momenta that the energy eigensates of the Bloch Hamiltonian are ill-defined. Note that the answers may depend on the numerical value of t_2.

(b) Similar to the QWZ model, find the gauge transformation between gauge I and II. Further more, make use of the result to determine the Chern numbers of the system in various phases.

Berry curvature

Remember in the 1D case, we discretize the momentum by introducing a large natural number N and define $p_1^{j_1} = j_1 \Delta$ with $\Delta = 2\pi/N$. For $\Delta \ll 1$, we have

$$\langle\, \psi(p_1^{j_1}) \,|\, \psi(p_1^{j_1+1}) \,\rangle \approx 1 - iA(p_1^{j_1})\,\Delta, \tag{80}$$

where $A(p_1^{j_1}) = i\,\langle\, \psi(p_1^{j_1}) \,|\, \partial_1 \psi(p_1^{j_1}) \,\rangle$. For 2D systems, we can define partial derivatives along p_1 and p_2. Hence, we have

$$\begin{aligned}
&\langle\, \psi(p_1^{j_1}, p_2^{j_2}) \,|\, \psi(p_1^{j_1+1}, p_2^{j_2}) \,\rangle \approx 1 - iA_1(p_1^{j_1}, p_2^{j_2})\,\Delta;\\
&\langle\, \psi(p_1^{j_1}, p_2^{j_2+1}) \,|\, \psi(p_1^{j_1}, p_2^{j_2}) \,\rangle \approx 1 - iA_2(p_1^{j_1}, p_2^{j_2})\,\Delta.
\end{aligned} \tag{81}$$

Here, $A_l(p_1^{j_1}, p_2^{j_2}) = i\,\langle\, \psi(p_1^{j_1}, p_2^{j_2}) \,|\, \partial_l \psi(p_1^{j_1}, p_2^{j_2}) \,\rangle$. Now, we are ready to introduce the discrete version of the Berry curvature:

$$\begin{aligned}
&\langle\, \psi(p_1^{j_1}, p_2^{j_2}) \,|\, \psi(p_1^{j_1+1}, p_2^{j_2}) \,\rangle + \langle\, \psi(p_1^{j_1+1}, p_2^{j_2}) \,|\, \psi(p_1^{j_1+1}, p_2^{j_2+1}) \,\rangle\\
&- \langle\, \psi(p_1^{j_1+1}, p_2^{j_2+1}) \,|\, \psi(p_1^{j_1}, p_2^{j_2+1}) \,\rangle - \langle\, \psi(p_1^{j_1}, p_2^{j_2+1}) \,|\, \psi(p_1^{j_1}, p_2^{j_2}) \,\rangle\,,\\
&\approx -i\Delta\Big\{ A_1(p_1^{j_1}, p_2^{j_2}) + A_2(p_1^{j_1+1}, p_2^{j_2}) - A_1(p_1^{j_1}, p_2^{j_2+1}) - A_2(p_1^{j_1}, p_2^{j_2}) \Big\}\\
&\approx -i\Delta^2 \Big\{ F_{12}(p_1^{j_1}, p_2^{j_2}) \Big\}.
\end{aligned} \tag{82}$$

Here, $F_{12}(p_1^{j_1}, p_2^{j_2}) = \partial_1 A_2(p_1^{j_1}, p_2^{j_2}) - \partial_2 A_1(p_1^{j_1}, p_2^{j_2})$. Obviously, this is the discrete version of Stokes' Theorem:

$$\int_\Omega d^2\boldsymbol{p}\, F_{12}(\boldsymbol{p}) = \int_{\partial\Omega} \boldsymbol{A}(\boldsymbol{p}) \cdot d\boldsymbol{p}.$$

In 2D, $F_{12}(\boldsymbol{p}) = \boldsymbol{\nabla} \times \boldsymbol{A}(\boldsymbol{p})$. Since

$$\begin{cases} \boldsymbol{A}^{(n)}(\boldsymbol{p}) = i \langle\, \psi_n(\boldsymbol{p}) \,|\, \boldsymbol{\nabla}\psi_n(\boldsymbol{p}) \,\rangle\,, \\ \langle\, \psi_n(\boldsymbol{p}) \,|\, \boldsymbol{\nabla}\psi_n(\boldsymbol{p}) \,\rangle = -\langle\, \boldsymbol{\nabla}\psi_n(\boldsymbol{p}) \,|\, \psi_n(\boldsymbol{p}) \,\rangle\,, \end{cases}$$

it can be seen that $\langle\, \psi_n(\boldsymbol{p}) \,|\, \boldsymbol{\nabla}\psi_n(\boldsymbol{p}) \,\rangle$ is purely imaginary. As result, $\boldsymbol{A}^{(n)}(\boldsymbol{p})$ is real. Consequently, we have

$$\boldsymbol{A}^{(n)}(\boldsymbol{p}) = -\mathrm{Im}\Big\{ \langle\, \psi_n(\boldsymbol{p}) \,|\, \boldsymbol{\nabla}\psi_n(\boldsymbol{p}) \,\rangle \Big\}, \tag{83}$$

and thus

$$\boldsymbol{B}^{(n)}(\boldsymbol{p}) = -\epsilon_{kl}\mathrm{Im}\left(\partial_k \Big\{ \langle\, \psi_n(\boldsymbol{p}) \,|\, \partial_l\psi_n(\boldsymbol{p}) \,\rangle \Big\}\right) = -\epsilon_{kl}\mathrm{Im}\Big\{ \langle\, \partial_k\psi_n(\boldsymbol{p}) \,|\, \partial_l\psi_n(\boldsymbol{p}) \,\rangle \Big\}. \tag{84}$$

Inserting a complete set, we have

$$\boldsymbol{B}^{(n)}(\boldsymbol{p}) = -\mathrm{Im} \sum_{n' \neq n} \Big\{ \langle\, \boldsymbol{\nabla}\psi_n(\boldsymbol{p}) \,|\, \psi_{n'}(\boldsymbol{p}) \,\rangle \times \langle\, \psi_{n'}(\boldsymbol{p}) \,|\, \boldsymbol{\nabla}\psi_n(\boldsymbol{p}) \,\rangle \Big\}. \tag{85}$$

We omit the term with $n' = n$. It is vanishing, as $\langle\, \boldsymbol{\nabla}\psi_n(\boldsymbol{p}) \,|\, \psi_n(\boldsymbol{p}) \,\rangle$ is parallel to $\langle\, \psi_n(\boldsymbol{p}) \,|\, \boldsymbol{\nabla}\psi_n(\boldsymbol{p}) \,\rangle$. From

$$\Big\{ \boldsymbol{\nabla}\mathcal{H}(\boldsymbol{p}) \Big\} \,|\, \psi_n(\boldsymbol{p}) \,\rangle + \mathcal{H}(\boldsymbol{p}) \,|\, \boldsymbol{\nabla}\psi_n(\boldsymbol{p}) \,\rangle = \Big\{ \boldsymbol{\nabla}E_n(\boldsymbol{p}) \Big\} \,|\, \psi_n(\boldsymbol{p}) \,\rangle + E_n(\boldsymbol{p}) \,|\, \boldsymbol{\nabla}\psi_n(\boldsymbol{p}) \,\rangle\,, \tag{86}$$

we see that

$$\langle\, \psi_n(\boldsymbol{p}) \,|\, \boldsymbol{\nabla}\mathcal{H}(\boldsymbol{p}) \,|\, \psi_n(\boldsymbol{p}) \,\rangle + E_{n'}(\boldsymbol{p}) \langle\, \psi_{n'}(\boldsymbol{p}) \,|\, \boldsymbol{\nabla}\psi_n(\boldsymbol{p}) \,\rangle = E_n(\boldsymbol{p}) \langle\, \psi_{n'}(\boldsymbol{p}) \,|\, \boldsymbol{\nabla}\psi_n(\boldsymbol{p}) \,\rangle\,, \tag{87}$$

by taking inner product with $\langle\, \psi_{n'}(\boldsymbol{p}) \,|$ from the left. Substitute the above result back to eq. (88), we finally achieve

$$\boldsymbol{B}^{(n)}(\boldsymbol{p}) = -\mathrm{Im} \sum_{n' \neq n} \frac{\Big\{ \langle\, \psi_n(\boldsymbol{p}) \,|\, \boldsymbol{\nabla}\mathcal{H}(\boldsymbol{p}) \,|\, \psi_{n'}(\boldsymbol{p}) \,\rangle \times \langle\, \psi_{n'}(\boldsymbol{p}) \,|\, \boldsymbol{\nabla}\mathcal{H}(\boldsymbol{p}) \,|\, \psi_n(\boldsymbol{p}) \,\rangle \Big\}}{\Big\{ E_n(\boldsymbol{p}) - E_{n'}(\boldsymbol{p}) \Big\}^2}. \tag{88}$$

This is especially useful in the case of a two-band system. A direct consequence of this expression is that the sum of the Berry curvature over all the bands would be zero, since the cross product in the numerator is anti-symmetric with repect to n and n'.

QWZ model with the periodic boundary condition along the x direction

By imposing the periodic boundary condition along the x direction, we have $\boldsymbol{a}_{j_1,j_2} = \boldsymbol{a}_{j_2} e^{ipj_1}$ and $\boldsymbol{b}_{j_1,j_2} = \boldsymbol{b}_{j_2} e^{ipj_1}$. Here, $p = 2\pi k/N$, with $k = 0, \ldots, N-1$. As a result, the Hamiltonian of the QWZ model becomes

$$\begin{aligned} \boldsymbol{H}(p) = \sum_{j_2} \Big\{ & (\mu - \cos p) \boldsymbol{a}_{j_2}^\dagger \boldsymbol{a}_{j_2} - (\mu - \cos p) \boldsymbol{b}_{j_2}^\dagger \boldsymbol{b}_{j_2} - \sin p\, \boldsymbol{a}_{j_2}^\dagger \boldsymbol{b}_{j_2} - \sin p\, \boldsymbol{b}_{j_2}^\dagger \boldsymbol{a}_{j_2} \\ & - \frac{1}{2} \boldsymbol{a}_{j_2+1}^\dagger \boldsymbol{a}_{j_2} - \frac{1}{2} \boldsymbol{a}_{j_2}^\dagger \boldsymbol{a}_{j_2+1} + \frac{1}{2} \boldsymbol{b}_{j_2+1}^\dagger \boldsymbol{b}_{j_2} + \frac{1}{2} \boldsymbol{b}_{j_2}^\dagger \boldsymbol{b}_{j_2+1} \\ & + \frac{1}{2} \boldsymbol{b}_{j_2+1}^\dagger \boldsymbol{a}_{j_2} + \frac{1}{2} \boldsymbol{a}_{j_2}^\dagger \boldsymbol{b}_{j_2+1} - \frac{1}{2} \boldsymbol{a}_{j_2+1}^\dagger \boldsymbol{b}_{j_2} - \frac{1}{2} \boldsymbol{b}_{j_2}^\dagger \boldsymbol{a}_{j_2+1} \Big\}. \end{aligned} \tag{89}$$

After the separation of variables, the Schrodinger equation is given by

$$\begin{aligned} (\mu - \cos p) a_{j_2} - \sin p\, b_{j_2} - \frac{1}{2} \Big\{ a_{j_2+1} + a_{j_2-1} \Big\} + \frac{1}{2} \Big\{ b_{j_2+1} - b_{j_2-1} \Big\} = E a_{j_2}, \\ -(\mu - \cos p) b_{j_2} - \sin p\, a_{j_2} + \frac{1}{2} \Big\{ b_{j_2+1} + b_{j_2-1} \Big\} - \frac{1}{2} \Big\{ a_{j_2+1} - a_{j_2-1} \Big\} = E b_{j_2}. \end{aligned} \tag{90}$$

By taking the sum and difference of the above equation, we obtain

$$
\begin{aligned}
(\mu - \cos p)\tilde{b}_{j_2} - \tilde{b}_{j_2-1} &= \Big\{E - \sin p\Big\}\tilde{a}_{j_2}, \\
(\mu - \cos p)\tilde{a}_{j_2} - \tilde{a}_{j_2+1} &= \Big\{E + \sin p\Big\}\tilde{b}_{j_2}.
\end{aligned} \tag{91}
$$

Here, $\tilde{a}_{j_2} = a_{j_2} - b_{j_2}$ and $\tilde{b}_{j_2} = a_{j_2} + b_{j_2}$. It can be seen that this is equivalent to the Rice-Mele model with $v = (\mu - \cos p)$ and $w = -1$ and chemiccla potential $\mu_{\text{eff}} = \sin p$. Therefore, the system will have edge states if the following condition is satisfied:

$$
|\mu - \cos p| < 1. \tag{92}
$$

The energy of the edge states is given by

$$
E \approx \pm \sin p. \tag{93}
$$

Low-energy limit of the Topological Insulators

It is known that the Bloch Hamiltonian of the SSH model is given by

$$
\boldsymbol{\mathcal{H}}_{\text{SSH}}(p) = \begin{pmatrix} 0 & -v - we^{-ip} \\ -v - we^{ip} & 0 \end{pmatrix}. \tag{94}
$$

The band edge occurs when $p = p_0$, with $p_0 = \pi$. In this limit, $E = \pm|v - w|$. To consider the low-energy limit of the system, we expand p around p_0:

$$
\boldsymbol{\mathcal{H}}_{\text{SSH}}(p) \approx \begin{pmatrix} 0 & -v + w\left\{1 - i\delta p\right\} \\ -v + w\left\{1 + i\delta p\right\} & 0 \end{pmatrix}. \tag{95}
$$

If we replace $\delta p = -i\partial_x$, then we can approximate the difference equation by a differential equation:

$$
\begin{aligned}
\Big\{(w - v) - w\partial_x\Big\}\psi_B(x) &= E\psi_A(x), \\
\Big\{(w - v) + w\partial_x\Big\}\psi_A(x) &= E\psi_B(x).
\end{aligned} \tag{96}
$$

This is analogous to the Dirac equation in (1+1)D. Combining the above coupled differential equation, we have

$$
\begin{aligned}
\Big\{(v - w)^2 - w^2\partial_x^2\Big\}\psi_A(x) &= E^2\psi_A(x), \\
\Big\{(v - w)^2 - w^2\partial_x^2\Big\}\psi_B(x) &= E^2\psi_B(x).
\end{aligned} \tag{97}
$$

The zero-energy edge state is given by

$$
\begin{aligned}
\psi_A(x) &= \alpha_+ e^{\frac{(v-w)x}{w}} + \alpha_- e^{-\frac{(v-w)x}{w}}, \\
\psi_B(x) &= \beta_+ e^{\frac{(v-w)x}{w}} + \beta_- e^{-\frac{(v-w)x}{w}}.
\end{aligned} \tag{98}
$$

For a right semi-infinite SSH chain, when $v < w$ the wave function of remains bounded only if $\alpha_- = \beta_- = 0$. From the boundary condition that $\psi_B(0) = 0$, we see that $\beta_+ = 0$ and hence $\psi_B(x) = 0$. Meanwhile, $\psi_A(x) \propto e^{\frac{(v-w)x}{w}}$ is an edge state bound to the left boundary $x = 0$. On the other hand, when $v > w$, the wave function of can remain bounded only if $\alpha_+ = \beta_+ = 0$. Therefore, we see that there is no edge state. The result is consistent with what we achieved previously using the difference equation approach.

We can also apply this kind of analysis in the low-energy limit to the QWZ model. There we will see that the energy-momentum dispersion relation of the system is analogous to that of a massless particle. The Bloch Hamiltonian of the QWZ model is given by

$$\boldsymbol{\mathcal{H}}_{\mathrm{QWZ}}(p) = \begin{pmatrix} \mu - \cos p_1 - \cos p_2 & \sin p_1 - i \sin p_2 \\ \sin p_1 + i \sin p_2 & -\mu + \cos p_1 + \cos p_2 \end{pmatrix}. \tag{99}$$

In the low-energy limit, $|p_1|, |p_2| \ll 1$ and we have

$$\boldsymbol{\mathcal{H}}_{\mathrm{QWZ}}(p) \approx \begin{pmatrix} \mu - 2 + \frac{p_1^2+p_2^2}{2} & p_1 - ip_2 \\ p_1 + ip_2 & -\mu + 2 - \frac{p_1^2+p_2^2}{2} \end{pmatrix}. \tag{100}$$

Imposing periodic and open boundary conditions along the x and y directions, respectively, we let $\mid \psi(x,y) \rangle = e^{ipx} \begin{pmatrix} \psi_u(y) \\ \psi_d(y) \end{pmatrix}$. As a result, we have

$$\begin{aligned} &\Big\{\mu - 2 + \frac{p^2}{2}\Big\}\psi_u(y) - \frac{1}{2}\psi_u''(y) + p\psi_d(y) - \psi_d'(y) = E\psi_u(y), \\ &p\psi_u(y) + \psi_u'(y) + \Big\{-\mu + 2 - \frac{p^2}{2}\Big\}\psi_d(y) + \frac{1}{2}\psi_d''(y) = E\psi_d(y). \end{aligned} \tag{101}$$

To find the edge state, let's assume $\psi_u(y) = c_u e^{-\lambda y}, \psi_d(y) = c_d e^{-\lambda y}$, and hence

$$\begin{aligned} &\Big\{\mu - 2 + \frac{p^2 - \lambda^2}{2} - E\Big\}c_u + \Big\{p + \lambda\Big\}c_d = 0, \\ &\Big\{p - \lambda\Big\}c_u + \Big\{-\mu + 2 - \frac{p^2 - \lambda^2}{2} - E\Big\}c_d = 0. \end{aligned} \tag{102}$$

For non-trivial solutions to exist, the determinant form by the coefficients of c_u and c_d in the above equation must be vanishing:

$$E^2 - \Big\{\mu - 2 + \frac{p^2 - \lambda^2}{2}\Big\}^2 - \Big\{p^2 - \lambda^2\Big\} = 0. \tag{103}$$

This is a quartic equation in λ and it is obvious that the roots are given by $\pm\lambda_1$ and $\pm\lambda_2$ From the above equation, we conclude thta λ_1 and λ_2 satisfy the following relations:

$$\begin{cases} \lambda_1^2\lambda_2^2 = \Big(2\mu + p^2 - 4\Big)^2 - 4\Big(E^2 - p^2\Big), \\ \lambda_1^2 + \lambda_2^2 = 2\Big(2\mu + p^2 - 2\Big). \end{cases} \tag{104}$$

Solving eq. (103) directly, we see that

$$\lambda^2 = 2\mu - 2 + p^2 \pm \sqrt{2\mu - 3 + E^2}. \tag{105}$$

If the real part of λ_1 and λ_2 are both positive, the wave function remains finite in the limit that y goes to infinity only if

$$\begin{cases} \psi_u(y) = c_{u,1}e^{-\lambda_1 y} + c_{u,2}e^{-\lambda_2 y}, \\ \psi_d(y) = c_{d,1}e^{-\lambda_1 y} + c_{d,2}e^{-\lambda_2 y}. \end{cases}$$

It goes without saying that $c_{u,\alpha}$ and $c_{d,\alpha}$ are related by

$$\Big\{\mu - 2 + \frac{p^2 - \lambda_\alpha^2}{2} - E\Big\}c_{u,\alpha} + \Big\{p + \lambda_\alpha\Big\}c_{d,\alpha} = 0, \tag{106}$$

where $\alpha = 1, 2$. It can be seen from eq. (90) that the boundary condition at $y = 0$, should be $\psi_u(0) = 0$ and $\psi_d(0) = 0$. Hence,

$$\begin{cases} c_{u,1} + c_{u,2} = 0, \\ \left\{ \frac{(E+2-\mu)}{p+\lambda_1} - \frac{p-\lambda_1}{2} \right\} c_{u,1} + \left\{ \frac{(E+2-\mu)}{p+\lambda_2} - \frac{p-\lambda_2}{2} \right\} c_{u,2} = 0. \end{cases} \tag{107}$$

From the above equation, the condition that the determinant formed by the coefficent of $c_{u,1}$ and $c_{u,2}$ leads to

$$\lambda_1\lambda_2 + p(\lambda_1 + \lambda_2) + 2\mu - 4 - 2E + p^2 = 0. \tag{108}$$

Making use of eq. (104), we see that the dispersion relation of the edge states are given by $E = \pm p$. Substitute the result back to eq (105), we see that λ_1 and λ_2 indeed are both real.